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Abstract

We consider inverse scattering for the time-harmonic wave equation with first-order perturbation in two dimensions. This problem arises in particular in the acoustic tomography of moving fluid. We consider linearized and nonlinearized reconstruction algorithms for this problem of inverse scattering. Our nonlinearized reconstruction algorithm is based on the non-local Riemann–Hilbert problem approach. Comparisons with preceding results are given.

Key words: acoustic tomography, moving fluid, wave equation with first-order perturbation, inverse scattering, Riemann–Hilbert problem.

1 Introduction

We consider the equation

$$-\Delta\psi - 2iA(x)\nabla\psi + V(x)\psi = E\psi, \quad x = (x_1, x_2) \in \mathbb{R}^2,\ E > 0, \tag{1.1}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, $\nabla = (\partial_{x_1}, \partial_{x_2})$, $\partial_{x_k} = \partial/\partial x_k$, $k = 1, 2$, and $A = (A_1, A_2)$ and V are vector and scalar potentials on $\mathbb{R}^2$, respectively. In addition we assume that

$$\begin{array}{l} A_1,\ A_2 \textit{ and } V \textit{ are sufficiently regular functions on } \mathbb{R}^2 \\ \textit{with sufficient decay at infinity.} \end{array} \tag{1.2}$$

Equation (1.1) can be considered as a model equation for the time-harmonic $\exp(-i\omega t)$ acoustic pressure ψ in a two-dimensional moving fluid, see e.g. [RW], [RE]. In this setting

$$E = \left(\frac{\omega}{c_0}\right)^2, \quad A(x) = \frac{\omega}{c_0}u(x), \quad V(x) = \left(1 - n^2(x)\right)\left(\frac{\omega}{c_0}\right)^2, \tag{1.3}$$

where c_0 is a reference sound speed, $n(x)$ is a scalar index of refraction, $u(x)$ is a normalized fluid velocity vector.

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Equation (1.1) can be also considered as the two-dimensional Schrödinger equation at fixed energy E with magnetic potential A and electric potential v, where

$$\begin{aligned} V(x) &= A^2(x) - i\,\mathrm{div}\, A(x) + v(x) \\ &= A_1^2(x) + A_2^2(x) - i\left(\frac{\partial A_1(x)}{\partial x_1} + \frac{\partial A_2(x)}{\partial x_2}\right) + v(x), \end{aligned} \tag{1.4}$$

see e.g. [HN1], [ER2].

For equation (1.1) we consider the classical scattering solutions ψ^+ continuous and bounded on $\mathbb{R}^2$ with their first derivatives and specified by the following asymptotics as $|x| \to \infty$:

$$\begin{gathered} \psi^+(x,k) = e^{ikx} + C(|k|)\frac{e^{i|k||x|}}{|x|^{1/2}} f\left(k, |k|\frac{x}{|x|}\right) + o\left(\frac{1}{|x|^{1/2}}\right), \\ x \in \mathbb{R}^2, \quad k \in \mathbb{R}^2,\ k^2 = E, \quad C(|k|) = -\pi i \sqrt{2\pi} e^{-i\pi/4}|k|^{1/2}, \end{gathered} \tag{1.5}$$

where a priori unknown function $f = f(k,l)$, $k,\ l \in \mathbb{R}^2$, $k^2 = l^2 = E$, arising in (1.5) is the classical scattering amplitude for (1.1).

Given potentials A, V, to determine ψ^+ and f one can use the following Lippmann–Schwinger integral equation (1.6) and formula (1.8) (see e.g. [HN1]):

$$\begin{gathered} \psi^+(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^2} G^+(x-y,k)\times \\ \times\big(-2iA(y)\nabla_y\psi^+(y,k) + V(y)\psi^+(y,k)\big)\,dy, \end{gathered} \tag{1.6}$$

$$G^+(x,k) = -(2\pi)^{-2}\int\limits_{\mathbb{R}^2} \frac{e^{i\xi x}\,d\xi}{\xi^2 - k^2 - i0} = -\frac{i}{4}H_0^1(|x||k|), \tag{1.7}$$

where $x \in \mathbb{R}^2$, $k \in \mathbb{R}^2$, $k^2 = E$, H_0^1 is the Hankel function of the first kind;

$$f(k,l) = (2\pi)^{-2}\int\limits_{\mathbb{R}^2} e^{-ily}\big(-2iA(y)\nabla_y\psi^+(y,k) + V(y)\psi^+(y,k)\big)\,dy, \tag{1.8}$$

where $k \in \mathbb{R}^2$, $l \in \mathbb{R}^2$, $k^2 = l^2 = E$. Actually, we consider (1.6) and its differentiated version, where ∇ is applied to both sided of (1.6), as a system of linear integral equations for ψ^+ and $\nabla\psi^+$.

One can see that the scattering amplitude f for equation (1.1) at fixed E is defined on

$$M_E = \big\{k \in \mathbb{R}^2,\ l \in \mathbb{R}^2 \colon k^2 = l^2 = E\big\}, \quad E > 0. \tag{1.9}$$

Note that f on M_E is invariant with respect to the gauge transformations

$$\begin{aligned} &A \to A + \nabla\varphi, \\ &V \to V - i\Delta\varphi + (\nabla\varphi)^2 + 2A\nabla\varphi, \end{aligned} \tag{1.10}$$

where φ is a sufficiently regular function on $\mathbb{R}^2$ with sufficient decay at infinity, see e.g. [HN2], [ER2]. In addition, ψ^+ is transformed as

$$\psi^+ \to e^{-i\varphi}\psi^+ \tag{1.11}$$

with respect to (1.10).

In this work we consider the following inverse scattering problem for equation (1.1) under assumptions (1.2):

Problem 1.1. Given scattering amplitude f on M_E at fixed $E > 0$, find potentials A and V on $\mathbb{R}^2$ (at least approximately).

Problem 1.1 for the case when $A \equiv 0$ was studied, in particular, in [N1], [GM], [N3], [GN2], [N4], [BBMRS], [BAR] and in [N2], [Buk].

Problem 1.1 for the general case was considered, in particular, on page 457 of [N3] and in [X].

Problem 1.1 is also related with several other inverse problems for the Schrödinger equation in magnetic field (and for the time-harmonic wave equation with first-order perturbation) in dimension $d \geq 2$. Concerning these other inverse problems see [DKN], [Sh], [HN2], [Nor], [RW], [RE], [BBS], [ER1], [NSU], [A], [ER2], [Ni], [WY], [GT], [IY] and references therein.

Note that approximate finding A and V in Problem 1.1 means, in particular, finding modulo transformations (1.10). However, for real-valued A and V there is no gauge nonuniqueness (1.10) in Problem 1.1! In addition, A and V of formulas (1.3) (of moving fluid acoustics) are real if n is real.

In this work we are mainly motivated by applications to the acoustic tomography of moving fluid discribed in [Nor], [RW], [RE], [BBS]. Note that in their reconstruction results works [Nor], [RW], [RE], [BBS] proceed from near-field scattering data (e.g. from some near field information on ψ^+) instead of the scattering amplitude f. But it is also known that near-field scattering data can be transformed into values of f, see e.g. [Ber], [BBS].

Results of the present work can be described as follows:

In Section 2 we give formulas for solving Problem 1.1 in the Born approximation. To our knowledge these formulas were not yet given explicitely in the literature for the case when $A \not\equiv 0$. These formulas are proved in Section 5.

In Section 3 we give a nonlinearized reconstruction algorithm for Problem 1.1. For the case when $A \equiv 0$ this algorithm is reduced to the algorithm of [N4] and was implemented numerically in [BAR]. For the general case this algorithm can be also regarded as simplication and development of the algorithm mentioned on page 457 of [N3] and based on the Riemann–Hilbert–Manakov problem approach of [GN1], [N1]. A derivation of the reconstruction algorithm of Section 3 is given in Section 6.

In Section 4 we show that in the Born approximation the algorithm of Section 3 is reduced to formulas of Section 2. Related proofs are given in Section 7.

In a similar way with results of [NS] the reconstruction algorithm of Section 3 can be generalized to the multi-channel case. This generalization will be given in a subsequent work.

In the present work we are focused on approximate reconstructions for Problem 1.1, admitting stable numerical implementation. Issues related with theorems of uniqueness and examples of nonuniqueness for Problem 1.1 will be considered in a subsequent work.

2 Inverse scattering in the Born approximation

If $A = (A_1, A_2)$ and V satisfy (1.2) and are sufficiently small, then proceeding from (1.6), (1.8) we have the following Born approximation formulas for direct scattering:

$$\begin{gathered} \psi^+(x,k) \approx e^{ikx}, \quad \nabla\psi^+(x,k) \approx e^{ikx} ik, \\ f(k,l) \approx f^{\mathrm{lin}}(k,l), \end{gathered} \tag{2.1}$$

$$f^{\mathrm{lin}}(k,l) \stackrel{\mathrm{def}}{=\!=} (2\pi)^{-2} \int\limits_{\mathbb{R}^2} e^{i(k-l)x} \big(2kA(x) + V(x)\big)\, dx, \tag{2.2}$$

where x, k, $l \in \mathbb{R}^2$, $k^2 = l^2 = E$. In particular, formulas (2.1) can be specified as (2.14).

Note that f^{lin} on M_E is invariant with respect to the gauge transformations

$$A \to A + \nabla\varphi, \quad V \to V - i\Delta\varphi, \tag{2.3}$$

where φ is a sufficiently regular function on $\mathbb{R}^2$ with sufficient decay at infinity. This invariance follows from (2.2), (2.3), integrating by parts and using that $k^2 - l^2 = 0$. We consider (2.3) as a linearization of (1.10) for small A, V and φ.

In this section we consider the following linearized inverse scattering problem for equation (1.1) under assumptions (1.2):

Problem 2.1. Given linearized scattering amplitude f^{lin} on M_E at fixed $E > 0$, find potentials A and V on $\mathbb{R}^2$ (at least approximately).

Note that approximate finding A and V in Problem 2.1 means, in particular, finding modulo transformations (2.3). However, in a similar way with Problem 1.1, there is no gauge nonuniqueness (2.3) in Problem 2.1 for the case of real-valued A and V.

Problem 2.1 is a linearization of Problem 1.1.

To study Problem 2.1 it is convenient to introduce φ^{div}, $A^{\mathrm{div},0}$, $V^{\mathrm{div},0}$ and $\varphi^{\pm}$, $A^{\pm,0}$, $V^{\pm,0}$ such that:

$$\begin{gathered} \Delta\varphi^{\mathrm{div}}(x) = -\operatorname{div} A(x), \quad \varphi^{\mathrm{div}}(x) \to 0 \text{ as } |x| \to \infty, \\ A^{\mathrm{div},0}(x) = A(x) + \nabla\varphi^{\mathrm{div}}(x), \quad V^{\mathrm{div},0}(x) = V(x) - i\Delta\varphi^{\mathrm{div}}(x), \end{gathered} \tag{2.4}$$

where $x \in \mathbb{R}^2$;

$$\begin{gathered} \partial_z\varphi^-(x) = -\frac{1}{2}(A_1(x) - iA_2(x)), \quad \varphi^-(x) \to 0 \text{ as } |x| \to \infty, \\ A^{-,0}(x) = A(x) + \nabla\varphi^-(x), \quad V^{-,0}(x) = V(x) - i\Delta\varphi^-(x), \end{gathered} \tag{2.5}$$

$$\begin{gathered} \partial_{\bar z}\varphi^+(x) = -\frac{1}{2}(A_1(x) + iA_2(x)), \quad \varphi^+(x) \to 0 \text{ as } |x| \to \infty, \\ A^{+,0}(x) = A(x) + \nabla\varphi^+(x), \quad V^{+,0}(x) = V(x) - i\Delta\varphi^+(x), \end{gathered} \tag{2.6}$$

where

$$\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \partial_{\bar z} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{2.7}$$

One can see that

$$\operatorname{div} A^{\mathrm{div},0}(x) = 0, \quad A_1^{-,0}(x) - iA_2^{-,0}(x) = 0, \quad A_1^{+,0}(x) + iA_2^{+,0}(x) = 0,$$

where $x \in \mathbb{R}^2$, $A^{\pm,0} = (A_1^{\pm,0}, A_2^{\pm,0})$.

It is also convenient to transform formula (2.2) to the form

$$\begin{gathered} f^{\mathrm{lin}}(k,l) - f^{\mathrm{lin}}(-l,-k) = 2(k+l)\widehat{A}(k-l), \\ f^{\mathrm{lin}}(k,l) + f^{\mathrm{lin}}(-l,-k) = 2(k-l)\widehat{A}(k-l) + 2\widehat{V}(k-l), \end{gathered} \tag{2.8}$$

$$\widehat{A}(p) = (2\pi)^{-2} \int\limits_{\mathbb{R}^2} e^{ipx} A(x)\, dx, \quad \widehat{V}(p) = (2\pi)^{-2} \int\limits_{\mathbb{R}^2} e^{ipx} V(x)\, dx, \tag{2.9}$$

where $(k,l) \in M_E$, $p \in \mathbb{R}^2$.

Note that

$$\begin{aligned} (k,l) \in M_E &\implies k - l \in B_{2\sqrt{E}}, \\ p \in B_{2\sqrt{E}} &\implies p = k - l \text{ for some } (k,l) \in M_E, \end{aligned} \tag{2.10}$$

where

$$B_r = \{p \in \mathbb{R}^2 \colon |p| \le r\}, \quad r > 0. \tag{2.11}$$

We define

$$\begin{gathered} C^{N,\sigma}(\mathbb{R}^2) = \big\{u \in C^N(\mathbb{R}^2) \colon \|u\|_{N,\sigma} < +\infty\big\}, \\ \|u\|_{N,\sigma} = \max_{|n| \leqslant N} \sup_{x \in \mathbb{R}^2} (1 + |x|^2)^{\sigma/2} |\partial^n u(x)|, \quad N \in \mathbb{N} \cup \{0\},\ \sigma \geqslant 0, \end{gathered} \tag{2.12}$$

where $C^N(\mathbb{R}^2)$ is the space of N-times continously differentiable complex-valued functions on $\mathbb{R}^2$,

$$\partial^n = \partial_{x_1}^{n_1} \partial_{x_2}^{n_2}, \quad n = (n_1, n_2) \in \big(\mathbb{N} \cup \{0\}\big)^2, \quad |n| = n_1 + n_2. \tag{2.13}$$

Note that if A_1, A_2, $V \in C^{0,\sigma}(\mathbb{R}^2)$ for some $\sigma > 2$ and $\|A_j\|_{0,\sigma} \le q$, $j = 1, 2$, $\|V\|_{0,\sigma} \le q$, then

$$\begin{gathered} \psi^+(x,k) = e^{ikx} + O(q), \quad \nabla\psi^+(x,k) = e^{ikx} ik + O(q), \\ f(k,l) = f^{\mathrm{lin}}(k,l) + O(q^2) \quad \text{as } q \to +0, \end{gathered} \tag{2.14}$$

uniformly with repsect to x, k, $l \in \mathbb{R}^2$, $k^2 = l^2 = E$, at fixed $E > 0$.

Theorem 2.1. *Suppose that A_1, A_2, V are real-valued and A_1, A_2, $V \in C^{N,\sigma}(\mathbb{R}^2)$ for some $N \ge 3$, $\sigma > 2$. Then the following formulas for solving*

Problem 2.1 are valid:

$$\widehat{A}(k-l)=\frac{f^{lin}(k,l)-\overline{f^{lin}(l,k)}}{2}\frac{k-l}{|k-l|^2}+\frac{f^{lin}(k,l)-f^{lin}(-l,-k)}{2}\frac{k+l}{|k+l|^2},$$
$$\widehat{V}(k-l)=\frac{\overline{f^{lin}(l,k)}+f^{lin}(-l,-k)}{2}, \tag{2.15}$$

where $\widehat{A}$, $\widehat{V}$ *are defined by* (2.9) *and* $(k,l)\in M_E$*;*

$$A(x)=A_{appr}(x,E)+A_{err}(x,E),\quad x\in\mathbb{R}^2,\ E>0,$$
$$A_{appr}(x,E)=\int\limits_{|p|\le 2\sqrt{E}} e^{-ipx}\widehat{A}(p)\,dp,\quad A_{err}(x,E)=\int\limits_{|p|\ge 2\sqrt{E}} e^{-ipx}\widehat{A}(p)\,dp, \tag{2.16}$$
$$V(x)=V_{appr}(x,E)+V_{err}(x,E),\quad x\in\mathbb{R}^2,\ E>0,$$
$$V_{appr}(x,E)=\int\limits_{|p|\le 2\sqrt{E}} e^{-ipx}\widehat{V}(p)\,dp,\quad V_{err}(x,E)=\int\limits_{|p|\ge 2\sqrt{E}} e^{-ipx}\widehat{V}(p)\,dp, \tag{2.17}$$

where $\widehat{A}(p)$ *and* $\widehat{V}(p)$ *for* $|p|\le 2\sqrt{E}$ *are given in terms of* f^{lin} *on* M_E *according to* (2.10), (2.15) *and*

$$|A_{err,j}(x,E)|\le c_1(N,\sigma)\|A_j\|_{N,\sigma}E^{-\frac{N-2}{2}}, \tag{2.18}$$
$$|V_{err}(x,E)|\le c_1(N,\sigma)\|V\|_{N,\sigma}E^{-\frac{N-2}{2}}, \tag{2.19}$$

where $x\in\mathbb{R}^2$, $j=1,\ 2$, $A_{err}=(A_{err,1},A_{err,2})$, $E\ge\frac{1}{4}$ *and*

$$c_1(N,\sigma)=\frac{4}{(N-2)(\sigma-2)}. \tag{2.20}$$

Theorem 2.2. *Suppose that* A_1, A_2, $V\in C^{N,\sigma}(\mathbb{R}^2)$ *for some* $N\ge 4$ *and* $\sigma>2$. *Let* $A^{div,0}$, $V^{div,0}$ *be defined according to* (2.4)*. Then the following formulas for solving Problem 2.1 are valid:*

$$\widehat{A}^{div,0}(k-l)=\frac{f^{lin}(k,l)-f^{lin}(-l,-k)}{2}\frac{k+l}{|k+l|^2},$$
$$\widehat{V}^{div,0}(k-l)=\frac{f^{lin}(k,l)+f^{lin}(-l,-k)}{2}, \tag{2.21}$$

where $\widehat{A}^{div,0}$, $\widehat{V}^{div,0}$ *are the Fourier transforms of* $A^{div,0}$, $V^{div,0}$ *(see* (2.9)*) and* $(k,l)\in M_E$*;*

$$A^{div,0}(x)=A^{div,0}_{appr}(x,E)+A^{div,0}_{err}(x,E),\quad x\in\mathbb{R}^2,\ E>0, \tag{2.22}$$
$$A^{div,0}_{appr}(x,E)=\int\limits_{|p|\le 2\sqrt{E}} e^{-ipx}\widehat{A}^{div,0}(p)\,dp,\quad A^{div,0}_{err}(x,E)=\int\limits_{|p|\ge 2\sqrt{E}} e^{-ipx}\widehat{A}^{div,0}(p)\,dp,$$
$$V^{div,0}(x)=V^{div,0}_{appr}(x,E)+V^{div,0}_{err}(x,E),\quad x\in\mathbb{R}^2,\ E>0, \tag{2.23}$$
$$V^{div,0}_{appr}(x,E)=\int\limits_{|p|\le 2\sqrt{E}} e^{-ipx}\widehat{V}^{div,0}(p)\,dp,\quad V^{div,0}_{err}(x,E)=\int\limits_{|p|\ge 2\sqrt{E}} e^{-ipx}\widehat{V}^{div,0}(p)\,dp,$$

where $\widehat{A}^{div,0}(p)$ *and* $\widehat{V}^{div,0}(p)$ *for* $|p| \le 2\sqrt{E}$ *are given in terms of* f^{lin} *on* M_E *according to* (2.10), (2.21) *and*

$$|A^{div,0}_{err,j}(x,E)| \le (1+\sqrt{2})c_1(N,\sigma)\|A\|_{N,\sigma}E^{-\frac{N-2}{2}}, \tag{2.24}$$

$$|V^{div,0}_{err}(x,E)| \le c_1(N,\sigma)\Big(\|V\|_{N,\sigma}E^{-\frac{N-2}{2}} + \sqrt{2}\|A\|_{N,\sigma}E^{-\frac{N-3}{2}}\Big), \tag{2.25}$$

$$\|A\|_{N,\sigma} = \max\big(\|A_1\|_{N,\sigma}, \|A_2\|_{N,\sigma}\big), \tag{2.26}$$

where $x \in \mathbb{R}^2$, $j = 1,\ 2$, $E \ge \frac{1}{4}$, $A^{div,0}_{err} = (A^{div,0}_{err,1}, A^{div,0}_{err,2})$ *and* $c_1(N,\sigma)$ *is defined by* (2.20). *Furthermore, if* $\operatorname{div} A = 0$ *then* $A^{div,0} = A$, $V^{div,0} = V$.

Theorem 2.3. *Suppose that* A_1, A_2, $V \in C^{N,\sigma}(\mathbb{R}^2)$ *for some* $N \ge 4$ *and* $\sigma > 2$. *Let* $A^{\pm,0}$, $V^{\pm,0}$ *be defined according to* (2.5)–(2.6). *Then the following formulas for solving Problem 2.1 are valid:*

$$\begin{gathered}\widehat{A}^{\pm,0}_1(k-l) = \frac{1}{2}\frac{f(k,l)-f(-l,-k)}{k_1+l_1\pm i(k_2+l_2)}, \quad \widehat{A}^{\pm,0}_2(k-l) = \pm i\widehat{A}^{\pm,0}_1(k-l),\\ \widehat{V}^{\pm,0}(k-l) = \frac{(l_1\pm il_2)f(k,l)+(k_1\pm ik_2)f(-l,-k)}{k_1+l_1\pm i(k_2+l_2)},\end{gathered} \tag{2.27}$$

where $\widehat{A}^{\pm,0}$, $\widehat{V}^{\pm,0}$ *are the Fourier transforms of* $A^{\pm,0}$, $V^{\pm,0}$ *(see* (2.9)*) and* $(k,l) \in M_E$*;*

$$A^{\pm,0}(x) = A^{\pm,0}_{appr}(x,E) + A^{\pm,0}_{err}(x,E), \quad x \in \mathbb{R}^2,\ E > 0, \tag{2.28}$$

$$A^{\pm,0}_{appr}(x,E) = \int\limits_{|p|\le 2\sqrt{E}} e^{-ipx}\widehat{A}^{\pm,0}(p)\,dp, \quad A^{\pm,0}_{err}(x,E) = \int\limits_{|p|\ge 2\sqrt{E}} e^{-ipx}\widehat{A}^{\pm,0}(p)\,dp,$$

$$V^{\pm,0}(x) = V^{\pm,0}_{appr}(x,E) + V^{\pm,0}_{err}(x,E), \quad x \in \mathbb{R}^2,\ E > 0, \tag{2.29}$$

$$V^{\pm,0}_{appr}(x,E) = \int\limits_{|p|\le 2\sqrt{E}} e^{-ipx}\widehat{V}^{\pm,0}(p)\,dp, \quad V^{\pm,0}_{err}(x,E) = \int\limits_{|p|\ge 2\sqrt{E}} e^{-ipx}\widehat{V}^{\pm,0}(p)\,dp,$$

where $\widehat{A}^{\pm,0}(p)$ *and* $\widehat{V}^{\pm,0}(p)$ *for* $|p| \le 2\sqrt{E}$ *are given in terms of* f^{lin} *on* M_E *according to* (2.10), (2.27) *and*

$$|A^{\pm,0}_{err,j}(x,E)| \le (1+\sqrt{2})c_1(N,\sigma)\|A\|_{N,\sigma}E^{-\frac{N-2}{2}}, \tag{2.30}$$

$$|V^{\pm,0}_{err}| \le c_1(N,\sigma)\Big(\|V\|_{N,\sigma}E^{-\frac{N-2}{2}} + \sqrt{2}\|A\|_{N,\sigma}E^{-\frac{N-3}{2}}\Big), \tag{2.31}$$

where $x \in \mathbb{R}^2$, $j = 1,\ 2$, $A^{\pm,0}_{err} = (A^{\pm,0}_{err,1}, A^{\pm,0}_{err,2})$, $\|A\|_{N,\sigma}$ *is defined by (2.23) and* $c_1(N,\sigma)$ *is given by* (2.20). *Furthermore, if* $A_1 \pm iA_2 = 0$ *then* $A = A^{\pm,0}$, $V = V^{\pm,0}$.

Theorems 2.1–2.3 are proved in Section 5.

3 Nonlinearized inverse scattering

3.1. Some notations. To study Problem 1.1 it is convenient to introduce φ^{div}, A^{div}, V^{div} and $\varphi^{\pm}$, $A^{\pm}$, $V^{\pm}$, where φ^{div} and $\varphi^{\pm}$ are defined according to (2.4)–(2.6) and

$$\begin{aligned} A^{\mathrm{div}} &= A + \nabla\varphi^{\mathrm{div}}, \quad V^{\mathrm{div}} = V - i\Delta\varphi^{\mathrm{div}} + (\nabla\varphi^{\mathrm{div}})^2 + 2A\nabla\varphi^{\mathrm{div}}, \\ A^{\pm} &= A + \nabla\varphi^{\pm}, \quad V^{\pm} = V - i\Delta\varphi^{\pm} + (\nabla\varphi^{\pm})^2 + 2A\nabla\varphi^{\pm}. \end{aligned} \tag{3.1}$$

In this section we give a nonlinearized algorithm for approximate finding $A^{\pm}$, $V^{\pm}$ and A^{div}, V^{div} on $\mathbb{R}^2$ from f on M_E. This algorithm takes into account multiple scattering effects and can be regarded as a nonlinear version of formulas for $A^{\pm,0}_{\mathrm{appr}}$, $V^{\pm,0}_{\mathrm{appr}}$, $A^{\mathrm{div},0}_{\mathrm{appr}}$, $V^{\mathrm{div},0}_{\mathrm{appr}}$ of (2.28), (2.29), (2.22), (2.23).

It is convenient to use the following notations:

$$z = x_1 + ix_2, \quad \bar z = x_1 - ix_2, \tag{3.2}$$

$$\lambda = E^{-1/2}(k_1 + ik_2), \quad \lambda' = E^{-1/2}(l_1 + il_2), \tag{3.3}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $k = (k_1, k_2) \in \Sigma_E$, $l = (l_1, l_2) \in \Sigma_E$,

$$\Sigma_E = \big\{m = (m_1, m_2) \in \mathbb{C}^2 \colon m_1^2 + m_2^2 = E\big\}, \quad E > 0. \tag{3.4}$$

In these notations

$$k_1 = \frac{1}{2}E^{1/2}(\lambda + \lambda^{-1}), \quad k_2 = \frac{i}{2}E^{1/2}(\lambda^{-1} - \lambda), \tag{3.5}$$

$$l_1 = \frac{1}{2}E^{1/2}(\lambda' + \lambda'^{-1}), \quad l_2 = \frac{i}{2}E^{1/2}(\lambda'^{-1} - \lambda'), \tag{3.6}$$

$$\exp(ikx) = \exp\Big(\frac{i}{2}E^{1/2}(\lambda\bar z + \lambda^{-1}z)\Big), \tag{3.7}$$

where λ, $\lambda' \in \mathbb{C}\setminus\{0\}$, $z \in \mathbb{C}^2$, k, $l \in \Sigma_E$.

In addition, using formulas (1.9), (3.3), (3.4), (3.5), (3.6) one can see that

$$\begin{gathered} \Sigma_E \cong \mathbb{C}\setminus\{0\}, \\ \Sigma_E \cap \mathbb{R}^2 = \mathbb{S}^1_{\sqrt{E}} \cong T, \\ M_E \cong T \times T, \end{gathered} \tag{3.8}$$

where

$$\begin{gathered} \mathbb{S}^1_r = \big\{m \in \mathbb{R}^2 \colon |m| = r\big\}, \quad r > 0, \\ T = \big\{\lambda \in \mathbb{C} \colon |\lambda| = 1\big\}. \end{gathered} \tag{3.9}$$

In addition, the functions ψ^+, f of (1.5)–(1.8) can be written as

$$\psi^+ = \psi^+(z, \lambda, E), \quad f = f(\lambda, \lambda', E), \tag{3.10}$$

where λ, $\lambda' \in T$, $z \in \mathbb{C}$, $E > 0$.

3.2. Reconstruction algorithm. Our nonlinearized algorithm for approximate finding $A^{\pm}$, $V^{\pm}$ and A^{div}, V^{div} on $\mathbb{R}^2$ from f on M_E has the following scheme

$$f \longrightarrow h_{\pm} \longrightarrow \mu^{+} \longrightarrow \mu_{\pm} \longrightarrow A^{\pm}_{\mathrm{appr}}, V^{\pm}_{\mathrm{appr}} \longrightarrow A^{\mathrm{div}}_{\mathrm{appr}}, V^{\mathrm{div}}_{\mathrm{appr}} \tag{3.11}$$

and consists of the following steps:

Step 1. Find functions $h_{\pm}(\lambda, \lambda', E)$, λ, $\lambda' \in T$, from the following linear integral equations:

$$h_{\pm}(\lambda, \lambda', E) - \pi i \int\limits_T h_{\pm}(\lambda, \lambda'', E) \chi \left(\pm i \left[\frac{\lambda}{\lambda''} - \frac{\lambda''}{\lambda} \right] \right) \times \\ \times f(\lambda'', \lambda', E)\, |d\lambda''| = f(\lambda, \lambda', E), \tag{3.12}$$

where

$$\chi(s) = \begin{cases} 1 & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases} \tag{3.13}$$

Step 2. Solve the following linear integral equation for $\mu^{+}(z, \lambda, E)$, $z \in \mathbb{C}$, $\lambda \in T$, $E > 0$:

$$\mu^{+}(z, \lambda, E) + \int\limits_T B(\lambda, \lambda', z, E) \mu^{+}(z, \lambda', E)\, |d\lambda'| = 1, \tag{3.14}$$

where

$$B(\lambda, \lambda', z, E) = \frac{1}{2} \int\limits_T h_{-}(\zeta, \lambda', z, E) \chi \left(-i \left[\frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta} \right] \right) \frac{d\zeta}{\zeta - \lambda(1-0)} - \\ - \frac{1}{2} \int\limits_T h_{+}(\zeta, \lambda', z, E) \chi \left(i \left[\frac{\zeta}{\lambda'} - \frac{\lambda'}{\zeta} \right] \right) \frac{d\zeta}{\zeta - \lambda(1+0)}, \tag{3.15}$$

$$h_{\pm}(\lambda, \lambda', z, E) \stackrel{\mathrm{def}}{=\!=} h_{\pm}(\lambda, \lambda', E) \times \\ \times \exp\Big(-i \frac{\sqrt{E}}{2} \big((\lambda - \lambda')\bar{z} + (\lambda^{-1} - \lambda'^{-1}) z \big) \Big), \tag{3.16}$$

and λ, $\lambda' \in T$, $z \in \mathbb{C}$, $E > 0$.

Step 3. Define functions $\mu_{\pm}(z, \lambda, E)$, $z \in \mathbb{C}$, $\lambda \in T$, $E > 0$, by formulas

$$\mu_{\pm}(z, \lambda, E) = \mu^{+}(z, \lambda, E) + \pi i \int\limits_T h_{\pm}(\lambda, \lambda', z, E) \times \\ \times \chi \left(\pm i \left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right] \right) \mu^{+}(z, \lambda', E)\, |d\lambda'|, \tag{3.17}$$

where functions $h_{\pm}(\lambda, \lambda', z, E)$ are given by (3.16) and χ is defined by (3.13).

Step 4. Functions $A^{\pm}_{\mathrm{appr,j}}(x,E)$, $V^{\pm}_{\mathrm{appr}}(x,E)$, $x\in\mathbb{R}^2$, $j=1,\,2$, $E>0$, are defined by formulas

$$
\begin{gathered}
A^{-}_{\mathrm{appr},1}(x,E)=\frac{i}{4}a^{-}_{z}(z,E),\quad A^{-}_{\mathrm{appr},2}(x,E)=\frac{1}{4}a^{-}_{z}(z,E),\\
a^{-}_{z}(z,E)=4\partial_{\bar z}\ln\int\limits_{T}\mu_{+}(z,\zeta,E)\,|d\zeta|,\\
V^{-}_{\mathrm{appr}}(x,E)=\frac{\sqrt{E}}{\pi}\int\limits_{T}\partial_{z}\mu_{-}(z,\zeta,E)\,d\zeta,
\end{gathered}
\tag{3.18}
$$

and

$$
\begin{gathered}
A^{+}_{\mathrm{appr},1}(x,E)=\frac{i}{4}a^{+}_{\bar z}(z,E),\quad A^{+}_{\mathrm{appr},2}(x,E)=-\frac{1}{4}a^{+}_{\bar z}(z,E),\\
a^{+}_{\bar z}(z,E)=-4\partial_{z}\ln\int\limits_{T}\mu_{+}(z,\zeta,E)\,|d\zeta|,\\
V^{+}_{\mathrm{appr}}(z,E)=2i\sqrt{E}\partial_{\bar z}\Bigg(\int\limits_{T}\mu_{+}(z,\zeta,E)\frac{d\zeta}{\zeta^2}\Bigg/\int\limits_{T}\mu_{+}(z,\zeta,E)\frac{d\zeta}{\zeta}\Bigg),
\end{gathered}
\tag{3.19}
$$

where z is given by (3.2).

Step 5. Find $A^{\mathrm{div}}_{\mathrm{appr,j}}(x,E)$, $V^{\mathrm{div}}_{\mathrm{appr}}(x,E)$, $x\in\mathbb{R}^2$, $j=1,\,2$, $E>0$, from formulas

$$
\begin{gathered}
A^{\mathrm{div}}_{\mathrm{appr},1}(x,E)=\frac{i}{8}(a^{-}_{z}(z,E)+a^{+}_{\bar z}(z,E)),\\
A^{\mathrm{div}}_{\mathrm{appr},2}(x,E)=\frac{1}{8}(a^{-}_{z}(z,E)-a^{+}_{\bar z}(z,E)),\\
V^{\mathrm{div}}_{\mathrm{appr}}(x,E)=\frac{1}{2}\big(V^{-}_{\mathrm{appr}}(x,E)+V^{+}_{\mathrm{appr}}(z,E)\big)-\frac{1}{8}a^{-}_{z}(z,E)a^{+}_{\bar z}(z,E),
\end{gathered}
\tag{3.20}
$$

where z is defined by (3.2) and functions a^{-}_{z}, $a^{+}_{\bar z}$, $V^{\pm}_{\mathrm{appr}}$ are defined in (3.18), (3.19).

A derivation of this reconstruction algorithm is based on the method of the Riemann–Hilbert problem and on the $\bar\partial$-method. This derivation is given in Section 6.

For the case when $A\equiv 0$ this algorithm is reduced to the algorithm of [N4] for approximatie finding V on $\mathbb{R}^2$ from f on M_E. The algorithm of [N4] consists of the same aforementioned steps 1, 2, 3 and the formula $V_{\mathrm{appr}}=V^{-}_{\mathrm{appr}}$, where V^{-}_{appr} is defined in (3.18). This algorithm of [N4] was implemented numerically in [BAR].

For the general case this algorithm can be also regarded as simplication and development of the algorithm mentioned (in few lines) on page 457 of [N3]. Actually, in [N3] the part of the algorithm consisting in finding $\mu_{\pm}$ from $h_{\pm}$

is realized in a more complicated way. In addition, in [N3] the algorithm is mentioned for the case when

$$A_1 = \overline{A}_1, \quad A_2 = \overline{A}_2, \quad -2i \operatorname{div} \overline{A} + \overline{V} = V, \tag{3.21}$$

i.e. for the self-adjoint case, whereas this assumption is not necessary for the algorithm.

3.3. Properties of the algorithm. Let

$$\begin{aligned} \|u_1\|_{L^2(T)} &= \Big(\int\limits_T |u_1(\lambda)|^2 \, |d\lambda| \Big)^{1/2}, \\ \|u_2\|_{L^2(T^2)} &= \Big(\iint\limits_{T^2} |u_2(\lambda, \lambda')|^2 \, |d\lambda| \, |d\lambda'| \Big)^{1/2}, \quad T^2 = T \times T, \end{aligned} \tag{3.22}$$

where u_1 and u_2 are test functions on T and T^2, respectively.

Proposition 3.1. *Let $E > 0$ be fixed. Suppose that*

$$f \in L^2(T^2), \quad \|f\|_{L^2(T^2)} < \frac{1}{\pi}, \tag{3.23}$$

where $f = f(\lambda, \lambda', E)$. Then equation (3.12) is uniquely solvable for $h_\pm \in L^2(T^2)$ and

$$\|h_\pm\|_{L^2(T^2)} < \frac{\|f\|_{L^2(T^2)}}{1 - \pi \|f\|_{L^2(T^2)}}, \tag{3.24}$$

$$\|B\|_{L^2(T^2)} < \frac{2\pi \|f\|_{L^2(T^2)}}{1 - \pi \|f\|_{L^2(T^2)}}, \tag{3.25}$$

where B is defined by (3.15), (3.16) (at fixed z, E). In addition, if

$$\|f\|_{L^2(T^2)} < \frac{1}{3\pi}, \tag{3.26}$$

then $\|B\|_{L^2(T^2)} < 1$, equation (3.14), at fixed z, E, is uniquely solvable for $\mu^+ \in L^2(T)$ and

$$\|\mu^+\|_{L^2(T)} < \frac{(2\pi)^{1/2}}{1 - \|B\|_{L^2(T^2)}}, \quad \|\mu^+ - 1\|_{L^2(T)} < \frac{(2\pi)^{1/2} \|B\|_{L^2(T^2)}}{1 - \|B\|_{L^2(T^2)}}, \tag{3.27}$$

$$\|\mu_\pm - 1\|_{L^2(T)} < \frac{3\pi (2\pi)^{1/2} \|f\|_{L^2(T^2)}}{1 - 3\pi \|f\|_{L^2(T^2)}}, \tag{3.28}$$

where $\mu_\pm$ are defined by (3.17). In addition, at least, if

$$\|f\|_{L^2(T^2)} < \frac{1}{6\pi}, \tag{3.29}$$

then

$$\int_T \mu_+(z,\lambda,E)\,|d\lambda| \neq 0 \quad \textit{for all } z\in\mathbb{C}, \tag{3.30}$$

and $A^{\pm}_{appr,j}$*,* $A^{div}_{appr,j}$*,* $j=1,\,2$*, as well as* $V^{\pm}_{appr}$*,* V^{div}_{appr} *are bounded on* $\mathbb{R}^2$*.*

Proposition 3.1 is based on solving the linear integral equations (3.12) and (3.14) by the method of successive approximations in $L^2(T^2)$ and $L^2(T)$, respectively, and on standard estimates of L^2-analysis for B, $h_\pm$, $\mu_\pm$ of (3.15), (3.16), (3.17) and for the integral of (3.30).

Note that assumptions (3.23), (3.26), (3.29) of Proposition 3.1 are only some surplus sufficient conditions on f for unique solvability of integral equations (3.12), (3.14), fulfilment of (3.30) and for boundedness of $A^{\mathrm{div}}_{\mathrm{appr}}$, $V^{\mathrm{div}}_{\mathrm{appr}}$.

Theorem 3.1. *Let* $f\in L^2(T^2)$ *at fixed* $E>0$*. Suppose that* f *satisfies (3.29) and is a smooth function on* T^2 *and* A^{div}_{appr}*,* V^{div}_{appr} *are constructed from* f *via the algorithm of Subsection (3.2). Then* $A^{div}_{appr,1}$*,* $A^{div}_{appr,2}$*,* V^{div}_{appr} *are bounded functions on* $\mathbb{R}^2$*, decaying at infinity. In addition,* f *is the scattering amplitude for equation (1.1) with* $A=A^{div}_{appr}(x,E)$*,* $V=V^{div}_{appr}(x,E)$*.*

For simplicity one can assume that $f\in C^\infty(T^2)$ in Theorem 3.1. However, very limited smoothness of f is already sufficient. As regards to smoothness of $A^{\mathrm{div}}_{\mathrm{appr},1}$, $A^{\mathrm{div}}_{\mathrm{appr},2}$, $V^{\mathrm{div}}_{\mathrm{appr}}$ of Theorem 3.1 (which are complex-valued, in general), these functions are real-analytic functions of $x\in\mathbb{R}^2$. In addition, it is just for simplicity that we assume (3.29) in Theorem 3.1.

The proof of Theorem 3.1 is similar to the proof of Theorem 9.2 of [N3] for the case when $A\equiv 0$. Results of this type go back to [N1]. In the present work restrictions in time prevent us from proving Theorem 3.1 in details.

Finally, suppose that $f=f(\lambda,\lambda',E)$ is the scattering amplitude for equation (1.1) under assumptions (1.2) and that $A^{\mathrm{div}}_{\mathrm{appr}}=A^{\mathrm{div}}_{\mathrm{appr}}(x,E)$, $V^{\mathrm{div}}_{\mathrm{appr}}=V^{\mathrm{div}}_{\mathrm{appr}}(x,E)$ are constructed from f via the algorithm of Subsection 3.2 at fixed E. In the present work restrictions on time prevent us from obtaining estimates for $A^{\mathrm{div}}-A^{\mathrm{div}}_{\mathrm{appr}}(\cdot,E)$ and $V^{\mathrm{div}}-V^{\mathrm{div}}_{\mathrm{appr}}(\cdot,E)$ for sufficiently large E. For the linearized case such error estimates are given by formulas (2.24), (2.25). For the nonlinearized case with $A\equiv 0$ such error estimates were given in [N4].

4 Reduction of the nonlinearized reconstruction algorithm to inversion formulas of the Born approximation

Suppose that we are given f on $M_E\cong T\times T=T^2$ at fixed E, where

$$f\in L^2(T^2),\quad \|f\|_{L^2(T^2)}\le\varepsilon. \tag{4.1}$$

Proposition 4.1. *Suppose that f satisfies* (4.1) *at fixed $E > 0$. Then, for $\varepsilon \to +0$, the nonlinearized reconstruction algorithm of Subsection 3.2 is reduced to the following formulas at fixed $E > 0$:*

$$\begin{gathered} A^{\pm}_{appr,j}(x,E) = \mathcal{A}^{\pm}_{appr,j}(x,E) + O(\varepsilon^2), \quad j = 1,2, \\ V^{\pm}_{appr}(x,E) = \mathcal{V}^{\pm}_{appr}(x,E) + O(\varepsilon^2), \end{gathered} \tag{4.2}$$

$$\begin{gathered} A^{div}_{appr}(x,E) = \mathcal{A}^{div}_{appr,j}(x,E) + O(\varepsilon^2), \quad j = 1,2, \\ V^{div}_{appr}(x,E) = \mathcal{V}^{div}_{appr}(x,E) + O(\varepsilon^2), \end{gathered} \tag{4.3}$$

where $O(\varepsilon^2)$ is considered in the uniform sense with respect to $x \in \mathbb{R}^2$ and where functions $\mathcal{A}^{\pm}_{appr,j}$, $\mathcal{V}^{\pm}_{appr}$, $j = 1$, 2, and $\mathcal{A}^{div}_{appr,j}$, $\mathcal{V}^{div}_{appr}$, $j = 1$, 2, are defined by the following linear formulas with respect to f:

$$\begin{gathered} \mathcal{A}^{-}_{appr,1}(x,E) = -\frac{i}{4}\sqrt{E}\int\limits_{T^2} \mathrm{sgn}\left(\frac{1}{i}\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right)(\lambda - \lambda')f(\lambda,\lambda',z,E)\,|d\lambda|\,|d\lambda'|, \\ \mathcal{A}^{-}_{appr,2}(x,E) = -i\mathcal{A}^{-}_{appr,1}(x,E), \\ \mathcal{V}^{-}_{appr}(x,E) = i\frac{E}{2}\int\limits_{T^2}(1 - \lambda\overline{\lambda'})\,\mathrm{sgn}\left(\frac{1}{i}\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right)f(\lambda,\lambda',z,E)\,|d\lambda|\,|d\lambda'|, \end{gathered} \tag{4.4}$$

$$\begin{gathered} \mathcal{A}^{+}_{appr,1}(x,E) = \frac{i}{4}\sqrt{E}\int\limits_{T^2} \mathrm{sgn}\left(\frac{1}{i}\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right)\overline{(\lambda - \lambda')}f(\lambda,\lambda',z,E)\,|d\lambda|\,|d\lambda'|, \\ \mathcal{A}^{+}_{appr,2}(x,E) = i\mathcal{A}^{+}_{appr,1}(x,E), \\ \mathcal{V}^{+}_{appr}(x,E) = -i\frac{E}{2}\int\limits_{T^2}(1 - \overline{\lambda}\lambda')\,\mathrm{sgn}\left(\frac{1}{i}\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right)f(\lambda,\lambda',z,E)\,|d\lambda|\,|d\lambda'|, \end{gathered} \tag{4.5}$$

$$\begin{gathered} \mathcal{A}^{div}_{appr,j}(x,E) = \frac{1}{2}\left(\mathcal{A}^{+}_{appr,j}(x,E) + \mathcal{A}^{-}_{appr,j}(x,E)\right), \quad j = 1,2, \\ \mathcal{V}^{div}_{appr}(x,E) = \frac{E}{2}\int\limits_{T^2}\left|\frac{1}{2i}\left(\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right)\right| f(\lambda,\lambda',z,E)\,|d\lambda|\,|d\lambda'|, \end{gathered} \tag{4.6}$$

where

$$\begin{gathered} f(\lambda,\lambda',z,E) \stackrel{def}{=\!=} f(\lambda,\lambda',E)\times \\ \times\exp\left(-i\frac{\sqrt{E}}{2}\left((\lambda - \lambda')\bar{z} + (\lambda^{-1} - \lambda'^{-1})z\right)\right), \end{gathered} \tag{4.7}$$

$\lambda \in T$, $\lambda' \in T$ and z, $\bar{z}$ are given by (3.2).

Proposition 4.2. *Suppose that A_1, A_2, $V \in C^{2,\sigma}(\mathbb{R}^2)$ for some $\sigma > 2$, where $C^{N,\sigma}(\mathbb{R}^2)$ is defined by (2.12). Let f^{lin} be defined by* (2.2) *and let $A^{div,0}_{appr,j}$, $V^{div,0}_{appr}$, $A^{\pm,0}_{appr,j}$, $V^{\pm,0}_{appr}$, $j = 1$, 2, be defined by* (2.21), (2.22), (2.23), (2.27), (2.28), (2.29) *in terms of f^{lin}. Suppose also that functions $\mathcal{A}^{div}_{appr,j}$, $\mathcal{V}^{div}_{appr}$, $\mathcal{A}^{\pm}_{appr,j}$, $\mathcal{V}^{\pm}_{appr}$, $j = 1$, 2, are given by* (4.6), (4.4), (4.5) *with $f = f^{lin}$. Then the following equalities* *are valid:*

$$A^{div,0}_{appr,j}(x,E) = \mathcal{A}^{div}_{appr,j}(x,E),$$
$$V^{div,0}_{appr}(x,E) = \mathcal{V}^{div}_{appr}(x,E), \tag{4.8}$$

$$A^{\pm,0}_{appr,j}(x,E) = \mathcal{A}^{\pm}_{appr,j}(x,E),$$
$$V^{\pm,0}_{appr}(x,E) = \mathcal{V}^{\pm}_{appr}(x,E), \tag{4.9}$$

where $x \in \mathbb{R}^2$, $j = 1,\ 2$, $E > 0$.

Propositions 4.1 and 4.2 are proved in Section 7.

5 Proofs of Theorems 2.1, 2.2, 2.3

Let us use the notations

$$\widehat{u}(p) = (2\pi)^{-2}\int\limits_{\mathbb{R}^2} e^{ipx}u(x)\,dx, \quad u_{\mathrm{err}}(x,E) = \int\limits_{|p|\ge 2\sqrt{E}} e^{-ipx}\widehat{u}(p)\,dp, \tag{5.1}$$

where $p \in \mathbb{R}^2$, $x \in \mathbb{R}^2$, $E > 0$.

Lemma 5.1. *Let* $u \in C^{N,\sigma}(\mathbb{R}^2)$, *where* $N \ge 3$, $\sigma > 2$. *Then the following formula holds:*

$$|u_{err}(x,E)| \le c_1(N,\sigma)\|u\|_{N,\sigma}E^{-\frac{N-2}{2}}, \tag{5.2}$$

where $x \in \mathbb{R}^2$, $E \ge 1/2$ *and* $c_1(N,\sigma)$ *is defined by* (2.20).

Proof of Lemma 5.1. We have that

$$\widehat{\partial^n u}(p) = (-ip_1)^{n_1}(-ip_2)^{n_2}\widehat{u}(p),$$

where ∂^n is defined in (2.13), $p = (p_1,p_2) \in \mathbb{R}^2$, $n = (n_1,n_2) \in (\mathbb{N}\cup\{0\})^2$, $|n| \le N$. Using this equality we obtain that

$$|\widehat{u}(p)| \le \frac{2^{N-1}}{\pi(\sigma-2)}\|u\|_{N,\sigma}\big(1+|p|^2\big)^{-\frac{N}{2}}$$

for each $p \in \mathbb{R}^2$, $|p| \ge 1$. Using the latter inequality we obtain (5.2). ☐

Proof of Theorem 2.1. Since potentials A and V are real-valued it follows from (2.8) that the following formula holds:

$$f^{\mathrm{lin}}(k,l) - \overline{f^{\mathrm{lin}}(l,k)} = 2(k-l)\widehat{A}(k-l), \tag{5.3}$$

where $(k,l) \in M_E$. We consider (2.8), (5.3) as a system of linear equations for finding $\widehat{A}(k-l)$ and $\widehat{V}(k-l)$. In addition, we use that $(k-l)(k+l) = 0$ for $(k,l) \in M_E$, i.e. that vectors $(k-l)$ and $(k+l)$ are orthogonal. As a result we obtain formulas (2.15).

Formulas (2.16), (2.17) can be regarded as definitions of A_{appr}, A_{err}, V_{appr}, V_{err}.

Estimates (2.18), (2.19) follow from Lemma 5.1. ☐

Lemma 5.2. *Suppose that A_1, $A_2 \in C^{N,\sigma}(\mathbb{R}^2)$, where $N \geq 4$, $\sigma > 2$. Let φ^{div} be the solution of* (2.4)*. Let $\widehat{\nabla\varphi}^{div}$, $(\nabla\varphi^{div})_{err}$, $\widehat{\Delta\varphi}^{div}$, $(\Delta\varphi^{div})_{err}$ be defined according to* (5.1)*. Then the following estimates hold:*

$$\begin{aligned}
&\big|(\partial_j\varphi^{div})_{err}(x,E)\big| \leq \sqrt{2}\, c_1(N,\sigma)\|A\|_{N,\sigma}E^{-\frac{N-2}{2}},\\
&\big|(\Delta\varphi^{div})_{err}(x,E)\big| \leq \sqrt{2}\, c_1(N,\sigma)\|A\|_{N,\sigma}E^{-\frac{N-3}{2}},
\end{aligned} \tag{5.4}$$

where $x \in \mathbb{R}^2$, $j = 1$, 2, $E \geq 1/4$, $\|A\|_{N,\sigma}$ is defined by (2.26) *and $c_1(N,\sigma)$ is defined by* (2.20)*.*

Proof of Lemma 5.2. The solution φ^{div} of (2.4) is given by:

$$\varphi^{\mathrm{div}}(x) = -i\int_{\mathbb{R}^2} e^{-ipx}\big(p\widehat{A}(p)\big)|p|^{-2}\,dp, \quad x \in \mathbb{R}^2. \tag{5.5}$$

Using (5.1), (5.5) we obtain that

$$\widehat{\nabla\varphi}^{\mathrm{div}}(p) = -p\big(p\widehat{A}(p)\big)|p|^{-2}, \quad \widehat{\Delta\varphi}^{\mathrm{div}}(p) = ip\widehat{A}(p), \tag{5.6}$$

where $p \in \mathbb{R}^2 \setminus \{0\}$. Formulas (5.6) imply the following inequalities:

$$\big|\widehat{\partial_j\varphi}^{\mathrm{div}}(p)\big| \leq \sqrt{2}\max_{k=1,2}|\widehat{A}_k(p)|, \quad \big|\widehat{\Delta\varphi}^{\mathrm{div}}(p)\big| \leq \sqrt{2}\,|p|\max_{k=1,2}|\widehat{A}_k(p)|, \tag{5.7}$$

where $p \in \mathbb{R}^2 \setminus \{0\}$, $j = 1$, 2.

We have that

$$\widehat{\partial^n A_j}(p) = (-ip_1)^{n_1}(-ip_2)^{n_2}\widehat{A}_j(p),$$

where $p = (p_1,p_2) \in \mathbb{R}^2$, $n = (n_1,n_2) \in (\mathbb{N}\cup\{0\})^2$, $|n| \leq N$. Using this equality we obtain that

$$|\widehat{A}_j(p)| \leq \frac{2^{N-1}}{\pi(\sigma-2)}\|A_j\|_{N,\sigma}(1+|p|^2)^{-\frac{N}{2}} \tag{5.8}$$

for $p \in \mathbb{R}^2$, $|p| \geq 1$, $j = 1$, 2. Formulas (5.7), (5.8) imply the estimates

$$\begin{aligned}
&\big|\widehat{\partial_j\varphi}^{\mathrm{div}}(p)\big| \leq \sqrt{2}\frac{2^{N-1}}{\pi(\sigma-2)}\|A\|_{N,\sigma}(1+|p|^2)^{-\frac{N}{2}},\\
&\big|\widehat{\Delta\varphi}^{\mathrm{div}}(p)\big| \leq \sqrt{2}\frac{2^{N-1}}{\pi(\sigma-2)}\|A\|_{N,\sigma}(1+|p|^2)^{-\frac{N-1}{2}},
\end{aligned} \tag{5.9}$$

where $p \in \mathbb{R}^2$, $|p| \geq 1$, $j = 1$, 2. Using (5.9) we obtain (5.4). ☐

Proof of Theorem 2.2. Taking into account invariance of f^{lin} with respect to transformations (2.3) and using (2.4), (2.8) we obtain the following equalities:

$$\begin{aligned}
(k-l)\widehat{A}^{\mathrm{div},0}(k-l) &= 0,\\
f^{\mathrm{lin}}(k,l) - f^{\mathrm{lin}}(-l,-k) &= 2(k+l)\widehat{A}^{\mathrm{div},0}(k-l),\\
f^{\mathrm{lin}}(k,l) + f^{\mathrm{lin}}(-l,-k) &= 2\widehat{V}^{\mathrm{div},0}(k-l),
\end{aligned} \tag{5.10}$$

where $(k,l)\in M_E$. Using (5.10) and orthogonality of vectors $(k-l)$ and $(k+l)$ we obtain (2.21).

Formulas (2.22), (2.23) can be regarded as definitions of $A^{\mathrm{div},0}_{\mathrm{appr}}$, $A^{\mathrm{div},0}_{\mathrm{err}}$, $V^{\mathrm{div},0}_{\mathrm{appr}}$, $V^{\mathrm{div},0}_{\mathrm{err}}$.

From (2.4), (2.22), (2.23) we derive

$$\begin{aligned} &A^{\mathrm{div},0}_{\mathrm{err,j}}(x,E)=(A_j)_{\mathrm{err}}(x,E)+(\partial_j\varphi^{\mathrm{div}})_{\mathrm{err}}(x,E), \quad j=1,2,\\ &V^{\mathrm{div},0}_{\mathrm{err}}(x,E)=V_{\mathrm{err}}(x,E)-i(\Delta\varphi^{\mathrm{div}})_{\mathrm{err}}(x,E), \end{aligned} \tag{5.11}$$

where $x\in\mathbb{R}^2$, $E>0$, and $(A_1)_{\mathrm{err}}$, $(A_2)_{\mathrm{err}}$, V_{err}, $(\nabla\varphi^{\mathrm{div}})_{\mathrm{err}}$, $(\Delta\varphi^{\mathrm{div}})_{\mathrm{err}}$ are defined according to (5.1).

From (5.11) using inequalities (5.2) for $(A_1)_{\mathrm{err}}$, $(A_2)_{\mathrm{err}}$, V_{err} and using inequalities (5.4) we obtain formulas (2.24), (2.25). $\square$

Lemma 5.3. *Suppose that A_1, $A_2\in C^{N,\sigma}(\mathbb{R}^2)$, where $N\geq 4$, $\sigma>2$. Let φ^-, φ^+ be the solutions of* (2.5), (2.6), *respectively. Let $\widehat{\nabla\varphi}^{\pm}$, $(\nabla\varphi^{\pm})_{err}$, $\widehat{\Delta\varphi}^{\pm}$, $(\Delta\varphi^{\pm})_{err}$ be defined according to* (5.1)*. Then the following estimates hold:*

$$\begin{aligned} &\big|(\partial_j\varphi^{\pm})_{err}(x,E)\big|\leq\sqrt{2}\,c_1(N,\sigma)\|A\|_{N,\sigma}E^{-\frac{N-2}{2}},\\ &\big|(\Delta\varphi^{\pm})_{err}(x,E)\big|\leq\sqrt{2}\,c_1(N,\sigma)\|A\|_{N,\sigma}E^{-\frac{N-3}{2}}, \end{aligned} \tag{5.12}$$

where $x\in\mathbb{R}^2$, $j=1,\,2$, $E\geq 1/4$, $\|A\|_{N,\sigma}$ is defined by (2.26) *and $c_1(N,\sigma)$ is defined by* (2.20)*.*

Proof of Lemma 2.3. The solutions $\varphi^{\pm}$ of (2.5), (2.6) are given by:

$$\varphi^{\pm}(x)=-i\int\limits_{\mathbb{R}^2}e^{-ipx}\frac{\widehat{A}_1(p)\pm i\widehat{A}_2(p)}{p_1\pm ip_2}dp, \quad x\in\mathbb{R}^2. \tag{5.13}$$

Using (5.1), (5.13) we obtain that

$$\widehat{\nabla\varphi}^{\pm}(p)=-\frac{\widehat{A}_1(p)\pm i\widehat{A}_2(p)}{p_1\pm ip_2}p, \quad \widehat{\Delta\varphi}^{\pm}(p)=i\frac{\widehat{A}_1(p)\pm i\widehat{A}_2(p)}{p_1\pm ip_2}|p|^2, \tag{5.14}$$

where $p\in\mathbb{R}^2\setminus\{0\}$. Formulas (5.14) imply the following inequalities:

$$\big|\widehat{\partial_j\varphi}^{\pm}(p)\big|\leq\sqrt{2}\max_{k=1,2}|\widehat{A}_k(p)|, \quad \big|\widehat{\Delta\varphi}^{\pm}(p)\big|\leq\sqrt{2}\,|p|\max_{k=1,2}|\widehat{A}_k(p)|, \tag{5.15}$$

where $p\in\mathbb{R}^2\setminus\{0\}$, $j=1,\,2$.

As in the proof of Lemma 5.2 we have estimates (5.8). Formulas (5.8), (5.15) imply the following estimates:

$$\begin{aligned} &\big|\widehat{\partial_j\varphi}^{\pm}(p)\big|\leq\sqrt{2}\frac{2^{N-1}}{\pi(\sigma-2)}\|A\|_{N,\sigma}(1+|p|^2)^{-\frac{N}{2}},\\ &\big|\widehat{\Delta\varphi}^{\pm}(p)\big|\leq\sqrt{2}\frac{2^{N-1}}{\pi(\sigma-2)}\|A\|_{N,\sigma}(1+|p|^2)^{-\frac{N-1}{2}}, \end{aligned} \tag{5.16}$$

where $p\in\mathbb{R}^2$, $|p|\geq 1$, $j=1,\,2$. Using (5.16) we obtain (5.12). $\square$

Proof of Theorem 2.3. Taking into account invariance of f^{lin} with respect to transformations (2.3) and using (2.5), (2.6), (2.8) we obtain the following equalities:

$$\begin{aligned} A_2^{\pm,0}(k-l) &= \pm i A_1^{\pm,0}(k-l), \\ f^{\mathrm{lin}}(k,l) - f^{\mathrm{lin}}(-l,-k) &= 2\big(k_1 + l_1 \pm i(k_2 + l_2)\big)\widehat{A}_1^{\pm,0}(k-l), \\ f^{\mathrm{lin}}(k,l) + f^{\mathrm{lin}}(-l,-k) &= 2\big(k_1 - l_1 \pm i(k_2 - l_2)\big)\widehat{A}_1^{\pm,0}(k-l) + 2\widehat{V}^{\pm,0}(k-l), \end{aligned} \tag{5.17}$$

where $(k,l) \in M_E$. Using (5.17) and orthogonality of vectors $(k-l)$ and $(k+l)$ we obtain (2.27).

Formulas (2.28), (2.29) can be regarded as definitions of $A_{\mathrm{appr}}^{\pm,0}$, $A_{\mathrm{err}}^{\pm,0}$, $V_{\mathrm{appr}}^{\pm,0}$, $V_{\mathrm{err}}^{\pm,0}$.

From (2.5), (2.6), (2.28), (2.29) we derive formulas

$$\begin{aligned} A_{\mathrm{err,j}}^{\pm,0}(x,E) &= (A_j)_{\mathrm{err}}(x,E) + (\nabla\varphi^{\pm})_{\mathrm{err}}(x,E), \quad j = 1,2, \\ V_{\mathrm{err}}^{\pm,0}(x,E) &= V_{\mathrm{err}}(x,E) - i(\Delta\varphi^{\pm})_{\mathrm{err}}(x,E), \end{aligned} \tag{5.18}$$

where $x \in \mathbb{R}^2$, $E > 0$, and $(A_1)_{\mathrm{err}}$, $(A_2)_{\mathrm{err}}$, V_{err}, $(\nabla\varphi^{\pm})_{\mathrm{err}}$, $(\Delta\varphi^{\pm})_{\mathrm{err}}$ are defined by (5.1).

From (5.18) using inequalities (5.2) for $(A_1)_{\mathrm{err}}$, $(A_2)_{\mathrm{err}}$, V_{err} and using inequalities (5.12) we obtain formulas (2.30), (2.31). $\square$

6 Derivation of the reconstruction algorithm of Section 3

6.1. Faddeev functions. For equation (1.1), under assumptions (1.2), we consider the Faddeev functions ψ, h (see e.g. [F1], [F2] and subsection 5.1 of [HN]):

$$\begin{aligned} \psi(x,k) &= e^{ikx}\mu(x,k), \\ \mu(x,k) &= 1 + \int_{\mathbb{R}^2} g(x-y,k)\times \\ &\times\big(-2iA(y)\nabla_y\mu(y,k) + \big(2A(y)k + V(y)\big)\mu(y,k)\big)\,dy, \end{aligned} \tag{6.1}$$

$$g(x,k) = -(2\pi)^{-2}\int_{\mathbb{R}^2} \frac{e^{i\xi x}}{\xi^2 + 2k\xi}d\xi, \tag{6.2}$$

where $x \in \mathbb{R}^2$, $k \in \Sigma_E \setminus \mathbb{R}^2$;

$$\begin{aligned} h(k,l) &= (2\pi)^{-2}\int_{\mathbb{R}^2} e^{i(k-l)y}\times \\ &\times\big(-2iA(y)\nabla_y\mu(y,k) + \big(2A(y)k + V(y)\big)\mu(y,k)\big)\,dy, \end{aligned} \tag{6.3}$$

where k, $l \in \Sigma_E \setminus \mathbb{R}^2$, $\operatorname{Im} k = \operatorname{Im} l$. Here (6.1) and its differentiated version, where ∇ is applied to both sides of (6.1), are considered as a system of linear integral equations for bounded μ and $\nabla\mu$, Σ_E is defined by (3.4).

We recall that ψ are «growing» solutions of (1.1), in the sense of [F1], parametrized by $k \in \Sigma_E \setminus \mathbb{R}^2$, and $G = e^{ikx}g$ is the Faddeev's Green function for the operator $\Delta + k^2$.

Equation (6.1) for μ and formula (6.3) for h are analogs in complex domain in k of equation (1.6) for ψ^+ and formula (1.8) for f.

Note that

$$k, l \in \Sigma_E \setminus \mathbb{R}^2,\ \operatorname{Im} k = \operatorname{Im} l \implies l = k \text{ or } l = -\overline{k}. \tag{6.4}$$

Therefore, the function h of (6.3) splits to the functions

$$a(k) = h(k, k), \quad b(k) = h(k, -\overline{k}), \quad k \in \Sigma_E \setminus \mathbb{R}^2. \tag{6.5}$$

Note also that a and b are invariant with respect to transformations (1.10) and ψ, μ are transformed as

$$\psi \to e^{-i\varphi}\psi, \quad \mu \to e^{-i\varphi}\mu \tag{6.6}$$

with respect to (1.10).

6.2. Analytic properties of ψ. Using notations of Subsection 3.1 coefficients A_1, A_2, V of equation (1.1), functions ψ^+, f of (1.6), (1.8) and functions ψ, μ, b of (6.1), (6.5) can be written as

$$\begin{gathered} A_1 = A_1(z), \quad A_2 = A_2(z), \quad V = V(z), \\ \psi^+ = \psi^+(z, \lambda, E), \quad f = f(\lambda, \lambda', E), \quad \lambda, \lambda' \in T, \\ \psi = \psi(z, \lambda, E),\ \mu = \mu(z, \lambda, E),\ b = b(\lambda, E),\ \lambda \in \mathbb{C} \setminus (T \cup 0), \end{gathered} \tag{6.7}$$

where $z \in \mathbb{C}$, $E > 0$.

It is known that the function ψ (or μ) has the following properties at fixed $z \in \mathbb{C}$ and $E > 0$ (see page 448 of [N3]):

$$\frac{\partial}{\partial \overline{\lambda}} \mu(z, \lambda, E) = r(\lambda, z, E)\mu\big(z, -\frac{1}{\overline{\lambda}}, E\big), \tag{6.8}$$

for $\lambda \in \mathbb{C} \setminus (T \cup 0)$, where

$$\begin{gathered} r(\lambda, z, E) = \exp\Big(-i\frac{\sqrt{E}}{2}\big(\lambda\overline{z} + \frac{z}{\lambda} + \overline{\lambda}z + \frac{\overline{z}}{\overline{\lambda}}\big)\Big)\times \\ \times\frac{\pi}{\overline{\lambda}}\operatorname{sgn}\big(\lambda\overline{\lambda} - 1\big)b(\lambda, E); \end{gathered} \tag{6.9}$$

$$\begin{aligned} \mu(z, \lambda, E) &= \mu_0^-(z) + o(1) \quad \text{for } \lambda \to \infty, \\ \mu(z, \lambda, E) &= \mu_0^+(z) + o(1) \quad \text{for } \lambda \to 0, \end{aligned} \tag{6.10}$$

where

$$\begin{gathered} \partial_{\bar{z}}\mu_0^+(z) = \frac{1}{2i}\big(A_1(z) + iA_2(z)\big)\mu_0^+(z), \\ \partial_z\mu_0^-(z) = \frac{1}{2i}\big(A_1(z) - iA_2(z)\big)\mu_0^-(z), \\ \mu_0^\pm(z) \to 1 \quad \text{as } z \to \infty; \end{gathered} \tag{6.11}$$

$$\psi_{\pm}(z,\lambda,E) = \psi^{+}(z,\lambda,E) + \pi i \int\limits_{T} h_{\pm}(\lambda,\lambda'',E)\chi\big(\pm i\left[\frac{\lambda}{\lambda''} - \frac{\lambda''}{\lambda}\right]\big)\times \\ \times\psi^{+}(z,\lambda'',E)\,|d\lambda''| \tag{6.12}$$

for $\lambda \in T$, where

$$\psi_{\pm}(z,\lambda,E) = \psi(z,\lambda(1\mp 0),E) = \exp\big(i\frac{\sqrt{E}}{2}\big(\lambda\bar z + \frac{z}{\lambda}\big)\big)\times \\ \times\mu_{\pm}(z,\lambda,E), \quad \mu_{\pm}(z,\lambda,E) = \mu(z,\lambda(1\pm 0),E), \quad \lambda\in T, \tag{6.13}$$

ψ^{+} is the function of (1.6), (6.7), $h_{\pm}$ are the functions related with the scattering amplitude f by equations (3.12), χ is defined by (3.13).

More precisely, equation (6.8) is fulfied if the system of linear equations for μ and $\nabla\mu$ related with (6.1) is uniquely solvable for $k = \big(k_1(\lambda,E),k_2(\lambda,E)\big)$ for fixed $\lambda\in\mathbb{C}\setminus(T\cup 0)$, where k_1, k_2 are given by (3.5), and relation (6.12) is fulfiled if the aforementioned system is uniquely solvable for $k = \big(k_1(\lambda(1\mp 0),E),k_2(\lambda(1\mp 0),E)\big)$ for fixed $\lambda\in T$. In particular, all these conditions are fulfiled if coefficients A_1, A_2, V of (1.1) are sufficiently small for fixed E.

6.3. Inverse scattering from f and b. Using the definitions of $\varphi^{\pm}$, $A^{\pm}$, $V^{\pm}$ of (2.5), (2.6), (3.1), the invariance of f and b with respect to (1.10) and formulas (1.11), (6.6) one can see that

for A, V transformed to $A^{\pm}$, $V^{\pm}$ (respectively) formulas (6.8)–(6.13) are fulfiled with $\mu_0^{\pm}\equiv 1$ (respectively). (6.14)

Properties (6.8)–(6.14) of ψ, μ yield the following approach to inverse scattering for equation (1.1) from f and b:

1. find ψ, μ satisfying (6.8)–(6.10), (6.12), (6.13) with a priori unknown ψ^{+} in (6.12), where $\mu_0^{-}\equiv 1$, $\mu(z,\cdot,E)\in C(\mathbb{C}\setminus T)$, $h_{\pm}$ are related with f by (3.12);

2. find A^{-}, V^{-} using that

$$A_1^{-}(z) - iA_2^{-}(z) = 0, \quad A_1^{-}(z) + iA_2^{-}(z) = 2i\partial_{\bar z}\ln\mu_0^{+}(z), \tag{6.15}$$

$$V^{-}(z)\psi(z,\lambda,E) = \big(4\partial_z\partial_{\bar z} + \\ +2i\big(A_1^{-}(z)+iA_2^{-}(z)\big)\partial_z + E\big)\psi(z,\lambda,E), \tag{6.16}$$

where $z\in\mathbb{C}$, $\lambda\in\mathbb{C}\setminus(T\cup 0)$.

Or alternatively:

1'. find ψ, μ satisfying (6.8)–(6.10), (6.12), (6.13) with a priori unknown ψ^{+} in (6.12), where $\mu_0^{+}\equiv 1$, $\mu(z,\cdot,E)\in C\big((\mathbb{C}\cup\infty)\setminus T\big)$, $h_{\pm}$ are related with f by (3.12);

2'. find A^+, V^+ using that

$$A_1^+(z) - iA_2^+(z) = 2i\partial_z \ln \mu_0^-(z), \quad A_1^+(z) + iA_2^+(z) = 0, \tag{6.17}$$

$$\begin{gathered} V^+(z)\psi(z,\lambda,E) = \big(4\partial_z\partial_{\bar z} + \\ +2i\big(A_1^+(z) - iA_2^+(z)\big)\partial_{\bar z} + E\big)\psi(z,\lambda,E), \end{gathered} \tag{6.18}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus (T \cup 0)$.

Note that (6.15) arises from (6.11) with $\mu_0^- \equiv 1$, (6.17) arises from (6.11) with $\mu_0^+ \equiv 1$, (6.16) and (6.18) arise from equation (1.1) for the Faddeev functions ψ of Subsections 6.1, 6.2 in the gauge setting related with A^-, V^- and A^+, V^+, respectively. In addition, ψ, μ, μ_0^+ of steps 1,2 and $\psi = \psi'$, $\mu = \mu'$, μ_0^- of steps 1', 2' are related by the formulas

$$\begin{gathered} \psi'(z,\lambda,E) = \big(\mu_0^+(z)\big)^{-1}\psi(z,\lambda,E), \\ \mu'(z,\lambda,E) = \big(\mu_0^+(z)\big)^{-1}\mu(z,\lambda,E), \quad \mu_0^-(z) = \big(\mu_0^+(z)\big)^{-1}, \end{gathered} \tag{6.19}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus (T \cup 0)$.

As soon as A, V are recovered as A^-, V^- or A^+, V^+, then these coefficients can be transformed from A^-, V^- or from A^+, V^+ to other possible gauge setting via (1.10) and, in particular, to A^{div}, V^{div} of (3.1).

Note that different ideas of the aforementioned approach to inverse scattering go back to [M], [ABF], [GN1], [GM], [N3], [N4]. In particular, finding ψ, μ of the aforementioned steps 1 or 1' for the case when $b \equiv 0$ at fixed E is reduced to solving a non-local Riemann–Hilbert problem for holomorphic functions, see Subsection 6.4. Such non-local Riemann–Hilbert problems go back to [M].

6.4. Inverse scattering with $b \equiv 0$ at fixed E. In the Born approximation at fixed E we have that

$$f(k,l) \approx f^{\mathrm{lin}}(k,l) = 2k\widehat{A}(k-l) + \widehat{V}(k-l), \quad (k,l) \in M_E, \tag{6.20}$$

$$b(k) \approx b^{\mathrm{lin}}(k) = 2k\widehat{A}(2\operatorname{Re} k) + \widehat{V}(2\operatorname{Re} k), \quad k \in \Sigma_E \setminus \mathbb{R}^2, \tag{6.21}$$

where $\widehat{A}$, $\widehat{V}$ are defined by (2.9). Here formula (6.20) is equivalent to the formulas for f of (2.1), (2.2) and formula (6.21) follows from (6.1), (6.3), (6.5) in a similar way that (6.20) follows from (1.6), (1.8). Note also that

$$k \in \Sigma_E \setminus \mathbb{R}^2 \implies 2\operatorname{Re} k \in \mathbb{R}^2 \setminus B_{2\sqrt{E}}, \quad E > 0, \tag{6.22}$$

where B_r is defined by (2.11).

Using (2.10), (6.20) and (6.21), (6.22) one can see that the expression for f^{lin} involves $\widehat{A}$, $\widehat{V}$ on $B_{2\sqrt{E}}$, only, and the expression for b^{lin} involves $\widehat{A}$, $\widehat{V}$ on $\mathbb{R}^2 \setminus B_{2\sqrt{E}}$, only, at fixed E. Further, using also (2.21)–(2.23), (2.27)–(2.29) one can see that the expressions for $A^{\pm,0}_{\mathrm{appr}}$, $V^{\pm,0}_{\mathrm{appr}}$, $A^{\mathrm{div},0}_{\mathrm{appr}}$, $V^{\mathrm{div},0}_{\mathrm{appr}}$ involve f^{lin}, only, and are independent of b^{lin} at fixed E.

In a similar way, in Section 3 in order to construct nonlinear analogs of $A^{\pm,0}_{\mathrm{appr}}$, $V^{\pm,0}_{\mathrm{appr}}$, $A^{\mathrm{div},0}_{\mathrm{appr}}$, $V^{\mathrm{div},0}_{\mathrm{appr}}$ we use inverse scattering of Subsection 6.3 without b or, in other words, with $b \equiv 0$ at fixed E.

In this case steps 1 and 1' of Subsection 6.3 consist in solving the following non-local Riemann–Hilbert problems for holomorphic functions:

1. find $\psi = \exp\big((i/2)\sqrt{E}(\lambda\bar{z} + z/\lambda)\big)\mu(z,\lambda,E)$, $z \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus (T \cup 0)$, satisfying (6.12), (6.13) with a priori unknown ψ^+ in (6.12), where

$$\begin{gathered}\frac{\partial}{\partial\bar{\lambda}}\mu(z,\lambda,E) = 0, \quad \lambda \in \mathbb{C} \setminus (T \cup 0),\\ \mu(z,\lambda,E) \to 1, \quad \text{as } \lambda \to \infty,\\ \mu(z,\cdot,E) \in C(\mathbb{C} \setminus T),\end{gathered} \tag{6.23}$$

or, alternatively:

1'. find $\psi = \exp\big((i/2)\sqrt{E}(\lambda\bar{z} + z/\lambda)\big)\mu(z,\lambda,E)$, $z \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus (T \cup 0)$, satisfying (6.12), (6.13) with a propri unknown ψ^+ in (6.12), where

$$\begin{gathered}\frac{\partial}{\partial\bar{\lambda}}\mu(z,\lambda,E) = 0, \quad \lambda \in \mathbb{C} \setminus (T \cup 0),\\ \mu(z,\lambda,E) \to 1, \quad \text{as } \lambda \to 0,\\ \mu(z,\cdot,E) \in C((\mathbb{C} \cup \infty) \setminus T).\end{gathered} \tag{6.24}$$

We recall that $h_\pm$ of (6.12) are related with f by (3.12). Actually, it is also assumed that $\mu(z,\cdot,E)$ admits continuous extension on T from its both sides.

Now due to considerations of Section 2 of [N4] we have that finding μ of step 1 is reduced to: (a) solving the linear integral equation (3.14) for $\mu^+(z,\cdot,E)$ on T, where $\psi^+ = \exp\big((i/2)\sqrt{E}(\lambda\bar{z} + z/\lambda)\big)\mu^+(z,\lambda,E)$, $z \in \mathbb{C}$, $\lambda \in T$, (b) finding $\mu_\pm(z,\cdot,E)$ on T by formulas (3.17) (or, in other words, by formulas (6.12) rewritten in terms of $\mu_\pm$ and μ^+), (c) finding $\mu(z,\cdot,E)$ on $C \setminus T$ by the Cauchy formulas:

$$\begin{aligned}\mu(z,\lambda,E) &= \frac{1}{2\pi i}\int\limits_T \frac{\mu_+(z,\zeta,E)}{\zeta - \lambda}d\zeta, \quad |\lambda| < 1,\\ \mu(z,\lambda,E) &= 1 - \frac{1}{2\pi i}\int\limits_T \frac{\mu_-(z,\zeta,E)}{\zeta - \lambda}d\zeta, \quad |\lambda| > 1,\end{aligned} \tag{6.25}$$

$z \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus T$.

In addition, due to (6.19), (6.25), finding $\psi = \psi'$, $\mu = \mu'$ of step 1' is reduced to finding ψ, μ of step 1 and to the formulas

$$\begin{gathered}\psi'(z,\lambda,E) = \big(\mu_0^+(z)\big)^{-1}\psi(z,\lambda,E), \quad \mu'(z,\lambda,E) = \big(\mu_0^+(z)\big)^{-1}\mu(z,\lambda,E),\\ \mu_0^+(z) = \frac{1}{2\pi i}\int\limits_T \frac{\mu_+(z,\zeta,E)}{\zeta}d\zeta,\end{gathered} \tag{6.26}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus (T \cup 0)$.

In addition, due to (6.25) we have that

$$\begin{aligned}\mu(z,\lambda,E) &= 1+\mu_1^-(z)\lambda^{-1}+O(\lambda^{-2}) \quad \text{as } \lambda\to\infty,\\ \mu(z,\lambda,E) &= \mu_0^+(z)+\mu_1^+(z)\lambda+O(\lambda^2) \quad \text{as } \lambda\to 0,\end{aligned} \tag{6.27}$$

where $z\in\mathbb{C}$, $\lambda\in\mathbb{C}\setminus(T\cup 0)$, $\mu_0^+(z)$ is given in (6.26),

$$\begin{aligned}\mu_1^+(z) &= \frac{1}{2\pi i}\int\limits_T \frac{\mu_+(z,\zeta,E)}{\zeta^2}d\zeta,\\ \mu_1^-(z) &= \frac{1}{2\pi i}\int\limits_T \mu_-(z,\zeta,E)\,d\zeta.\end{aligned} \tag{6.28}$$

Note that the non-local Riemann–Hilbert problems of steps 1 and 1' are better known in the literature (going back to [M]) for the case when relation (6.12) between ψ_+ and ψ_- on T with a priori unknown ψ^+ is given in the form

$$\psi_+(\lambda)=\psi_-(\lambda)+\int\limits_T \rho(\lambda,\lambda')\psi_-(\lambda')\,|d\lambda'|, \quad \lambda\in T, \tag{6.29}$$

see e.g. [M], [GN1], [GM], [N3]. Note also that in our case $h_\pm$ of (3.12), (6.12) are related with ρ of (6.29) by the following formulas and equations:

$$\begin{aligned}h_1(\lambda,\lambda',E) &= \chi\Big(i\Big[\frac{\lambda'}{\lambda}-\frac{\lambda}{\lambda'}\Big]\Big)h_+(\lambda,\lambda',E)-\\ &\quad -\chi\Big(-i\Big[\frac{\lambda'}{\lambda}-\frac{\lambda}{\lambda'}\Big]\Big)h_-(\lambda,\lambda',E),\\ h_2(\lambda,\lambda',E) &= \chi\Big(i\Big[\frac{\lambda'}{\lambda}-\frac{\lambda}{\lambda'}\Big]\Big)h_-(\lambda,\lambda',E)-\\ &\quad -\chi\Big(-i\Big[\frac{\lambda'}{\lambda}-\frac{\lambda}{\lambda'}\Big]\Big)h_+(\lambda,\lambda',E),\end{aligned} \tag{6.30}$$

$$\begin{aligned}&\rho(\lambda,\lambda',E)+\pi i\int\limits_T \rho(\lambda,\lambda'',E)\chi\Big(-i\Big[\frac{\lambda'}{\lambda''}-\frac{\lambda''}{\lambda'}\Big]\Big)\times\\ &\qquad\times h_1(\lambda'',\lambda',E)\,|d\lambda''| = -\pi i h_1(\lambda,\lambda',E),\\ &\rho(\lambda,\lambda',E)+\pi i\int\limits_T \chi\Big(i\Big[\frac{\lambda'}{\lambda''}-\frac{\lambda''}{\lambda'}\Big]\Big)\times\\ &\qquad\times h_2(\lambda'',\lambda',E)\,|d\lambda''| = -\pi i h_2(\lambda,\lambda',E),\end{aligned} \tag{6.31}$$

where λ, $\lambda'\in T$, see [N3].

Futher, due to results of [GN1], [N3] and of Proposition 3.1 of the present work we have that, at least under assumption (3.26), the non-local Riemann–Hilbert problems for ψ and for $\psi=\psi'$ of steps 1 and 1' are uniquely solvable and

$$\big(-4\partial_z\partial_{\bar z} + a_z^-(z)\partial_z + V^-(z)\big)\psi(z,\lambda,E) = E\psi(z,\lambda,E), \tag{6.32}$$

$$a_z^-(z) = 4\partial_{\bar z}\ln\mu_0^+(z), \quad V^-(z) = 2i\sqrt{E}\partial_z\mu_1^-(z), \tag{6.33}$$

$$\big(-4\partial_z\partial_{\bar z} + a_{\bar z}^+(z)\partial_{\bar z} + V^+(z)\big)\psi'(z,\lambda,E) = E\psi'(z,\lambda,E), \tag{6.34}$$

$$a_{\bar z}^+(z) = 4\partial_z\ln\frac{1}{\mu_0^+(z)}, \quad V^+(z) = 2i\sqrt{E}\partial_{\bar z}\frac{\mu_1^+(z)}{\mu_0^+(z)}, \tag{6.35}$$

where $z\in\mathbb{C}$, $\lambda\in\mathbb{C}\setminus(T\cup 0)$, μ_0^+, μ_1^-, μ_1^+ are the coefficients of (6.27). Here (6.32), (6.33) correspond to (6.15), (6.16) and (6.34), (6.35) correspond to (6.17), (6.18).

Formulas (3.18), (3.19) follow from (6.32)–(6.35) and the integral expression for μ_0^+, μ_1^-, μ_1^+ of (6.26), (6.28).

Finally, formulas (3.20) arise from considerations of the gauge transformations (1.10) between A^-, V^- and A^+, V^+ and A^{div}, V^{div} of (3.1). In particular, in these considerations we use that for equation (1.1) written as

$$\begin{gathered}\big(-4\partial_z\partial_{\bar z} + a_z\partial_z + a_{\bar z}\partial_{\bar z} + V\big)\psi = E\psi,\\ a_z = -2i(A_1+iA_2), \quad a_{\bar z} = -2i(A_1 - iA_2),\end{gathered} \tag{6.36}$$

the gauge transformations (1.10), (1.11) can be written as

$$\begin{aligned}&a_z\to a_z - 4i\partial_{\bar z}\varphi, \quad a_{\bar z}\to a_{\bar z} - 4i\partial_z\varphi,\\ &V\to V - 4i\partial_z\partial_{\bar z}\varphi + 4\partial_z\varphi\partial_{\bar z}\varphi + ia_z\partial_z\varphi + ia_{\bar z}\partial_{\bar z}\varphi,\\ &\psi\to e^{-i\varphi}\psi,\end{aligned} \tag{6.37}$$

and that the equations for φ^{div}, φ^-, φ^+ of (2.4)–(2.6), (3.1) can be written as

$$\begin{gathered}8i\partial_z\partial_{\bar z}\varphi^{\mathrm{div}} = \partial_z a_z + \partial_{\bar z}a_{\bar z}, \quad \varphi^{\mathrm{div}}(z)\to 0 \text{ as } |z|\to\infty,\\ 4i\partial_z\varphi^- = a_{\bar z}, \quad \varphi^-(z)\to 0 \text{ as } |z|\to\infty,\\ 4i\partial_{\bar z}\varphi^+ = a_z, \quad \varphi^+(z)\to 0 \text{ as } |z|\to\infty.\end{gathered} \tag{6.38}$$

7 Proofs of Propositions 4.1 and 4.2

Proof of Proposition 4.1. The method of successive approximations for solving (3.12) with repsect to $h_\pm\in L^2(T^2)$ and assumptions (4.1) imply that

$$h_\pm = f + O(\varepsilon^2), \quad \varepsilon\to +0, \tag{7.1}$$

where $O(\varepsilon^2)$ is considered in the sense of $\|\cdot\|_{L^2(T^2)}$.

Consider the following operators acting in $L^2(T)$

$$(C_\pm u)(\lambda) = \frac{1}{2\pi i}\int\limits_T \frac{u(\zeta)}{\zeta - \lambda(1\mp 0)}\,d\zeta, \quad \lambda\in T, \tag{7.2}$$

where u is a test function. We recall that

$$\|C_\pm u\|_{L^2(T)} \le \|u\|_{L^2(T)}. \tag{7.3}$$

Using (3.15), (3.16), (7.1), (7.3) and the equality

$$\begin{gathered}\left|\exp\Big(-i\frac{\sqrt{E}}{2}\big((\lambda-\lambda')\bar z+(\lambda^{-1}-\lambda'^{-1})z\big)\Big)\right| = 1,\\ \lambda,\lambda'\in T,\quad z\in\mathbb{C},\quad E>0,\end{gathered} \tag{7.4}$$

we obtain that

$$\begin{aligned} B(\lambda,\lambda',z,E) = \frac{1}{2}\int\limits_T f(\zeta,\lambda',z,E)\chi\Big(-i\left[\frac{\zeta}{\lambda'}-\frac{\lambda'}{\zeta}\right]\Big)\frac{d\zeta}{\zeta-\lambda(1-0)} - \\ -\frac{1}{2}\int\limits_T f(\zeta,\lambda',z,E)\chi\Big(i\left[\frac{\zeta}{\lambda'}-\frac{\lambda'}{\zeta}\right]\Big)\frac{d\zeta}{\zeta-\lambda(1+0)} + O(\varepsilon^2),\end{aligned} \tag{7.5}$$

where λ, $\lambda'\in T$, $z\in\mathbb{C}$, $f(\zeta,\lambda',z,E)$ is given by (4.7), $O(\varepsilon^2)$ is considered in the sense of $\|\cdot\|_{L^2(T^2)}$ and is uniform with respect to $z\in\mathbb{C}$.

From (3.14) we derive the following equalities:

$$\begin{aligned} &\partial_z\mu^+(z,\lambda,E) + \int\limits_T B(\lambda,\lambda',z,E)\partial_z\mu^+(z,\lambda',E)\,|d\lambda'| = \\ &\quad = -\int\limits_T \partial_z B(\lambda,\lambda',z,E)\mu^+(z,\lambda',E)\,|d\lambda'|,\\ &\partial_{\bar z}\mu^+(z,\lambda,E) + \int\limits_T B(\lambda,\lambda',z,E)\partial_{\bar z}\mu^+(z,\lambda',E)\,|d\lambda'| = \\ &\quad = -\int\limits_T \partial_{\bar z} B(\lambda,\lambda',z,E)\mu^+(z,\lambda',E)\,|d\lambda'|,\end{aligned} \tag{7.6}$$

where $\lambda\in T$, $z\in\mathbb{C}$, $E>0$.

The method of successive approximations for solving (3.14) and (7.6) with respect to $\mu_+\in L^2(T)$ and $\partial_z\mu_+$, $\partial_{\bar z}\mu_+\in L^2(T)$ and estimates (4.1), (7.5) imply that:

$$\begin{aligned} &\mu^+(z,\lambda,E) = 1+O(\varepsilon),\\ &\partial_z\mu^+(z,\lambda,E) = -\int\limits_T \partial_z B(\lambda,\lambda',z,E)\,|d\lambda'| + O(\varepsilon^2),\\ &\partial_{\bar z}\mu^+(z,\lambda,E) = -\int\limits_T \partial_{\bar z} B(\lambda,\lambda',z,E)\,|d\lambda'| + O(\varepsilon^2),\end{aligned} \tag{7.7}$$

where $z\in\mathbb{C}$, $\lambda\in T$, $O(\varepsilon)$, $O(\varepsilon^2)$ are considered in the sense of $\|\cdot\|_{L^2(T)}$ and are uniform with respect to $z\in\mathbb{C}$.

Using (3.17), (7.1), (7.7) we obtain that

$$\begin{aligned}
\mu_\pm(z,\lambda,E) &= 1 + O(\varepsilon),\\
\partial_z\mu_\pm(z,\lambda,E) &= -\int\limits_T \partial_z B(\lambda,\lambda',z,E)\,|d\lambda'| + \\
&\quad + \pi i \int\limits_T \partial_z f(\lambda,\lambda',z,E)\chi\Big(\pm i\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\Big)\,|d\lambda'| + O(\varepsilon^2),\\
\partial_{\bar z}\mu_\pm(z,\lambda,E) &= -\int\limits_T \partial_{\bar z} B(\lambda,\lambda',z,E)\,|d\lambda'| + \\
&\quad + \pi i \int\limits_T \partial_{\bar z} f(\lambda,\lambda',z,E)\chi\Big(\pm i\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\Big)\,|d\lambda'| + O(\varepsilon^2),
\end{aligned} \tag{7.8}$$

where $\lambda \in T$, $f(\lambda,\lambda',z,E)$ is given by (4.7) and $O(\varepsilon)$, $O(\varepsilon^2)$ are considered in the sense of $\|\cdot\|_{L^2(T)}$ and are uniform with repsect to $z \in \mathbb{C}$.

Using (3.18), (3.19), (7.8) we obtain that:

$$\begin{aligned}
V^-_{\mathrm{appr}}(x,E) &= -\frac{\sqrt{E}}{\pi}\int\limits_{T^2} \partial_z B(\lambda,\lambda',z,E)\,d\lambda\,|d\lambda'| + \\
&\quad + i\sqrt{E}\int\limits_{T^2} \partial_z f(\lambda,\lambda',z,E)\chi\Big(-i\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\Big)\,d\lambda\,|d\lambda'| + O(\varepsilon^2),\\
V^+_{\mathrm{appr}}(x,E) &= \frac{\sqrt{E}}{\pi}\int\limits_{T^2} \partial_{\bar z} B(\lambda,\lambda',z,E)\,\lambda^{-2}d\lambda\,|d\lambda'| - \\
&\quad - i\sqrt{E}\int\limits_{T^2} \partial_{\bar z} f(\lambda,\lambda',z,E)\chi\Big(i\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\Big)\,\lambda^{-2}d\lambda\,|d\lambda'| + O(\varepsilon^2),\\
a^-_z(z,E) &= \frac{2i}{\pi}\int\limits_{T^2} \partial_{\bar z} B(\lambda,\lambda',z,E)\,\lambda^{-1}d\lambda\,|d\lambda'| + \\
&\quad + 2\int\limits_{T^2} \partial_{\bar z} f(\lambda,\lambda',z,E)\chi\Big(i\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\Big)\,\lambda^{-1}d\lambda\,|d\lambda'| + O(\varepsilon^2),\\
a^+_{\bar z}(z,E) &= -\frac{2i}{\pi}\int\limits_{T^2} \partial_z B(\lambda,\lambda',z,E)\,\lambda^{-1}d\lambda\,|d\lambda'| - \\
&\quad - 2\int\limits_{T^2} \partial_z f(\lambda,\lambda',z,E)\chi\Big(i\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\Big)\,\lambda^{-1}d\lambda\,|d\lambda'| + O(\varepsilon^2),
\end{aligned} \tag{7.9}$$

where $x \in \mathbb{R}^2$, z is given by (3.2), and $O(\varepsilon^2)$ is considered in the uniform sense with respect to $x \in \mathbb{R}^2$.

Note that the following formulas hold for each $u \in L^2(T)$:

$$\begin{gathered}
\int_T (C_+u)(\lambda)\, d\lambda = 0, \quad \int_T (C_-u)(\lambda)\, d\lambda = -\int_T u(\lambda)\, d\lambda, \\
\int_T (C_+u)(\lambda)\, \frac{d\lambda}{\lambda} = \int_T u(\lambda)\, \frac{d\lambda}{\lambda}, \quad \int_T (C_-u)(\lambda)\, \frac{d\lambda}{\lambda} = 0, \\
\int_T (C_+u)(\lambda)\, \frac{d\lambda}{\lambda^2} = \int_T u(\lambda)\, \frac{d\lambda}{\lambda^2}, \quad \int_T (C_-u)(\lambda)\, \frac{d\lambda}{\lambda^2} = 0,
\end{gathered} \tag{7.10}$$

where $C_\pm$ are defined by (7.2).

Formulas (7.5), (7.9), (7.10) imply estimates

$$\begin{aligned}
V^-_{\mathrm{appr}}(x,E) &= i\sqrt{E}\int_{T^2} s(\lambda,\lambda')\partial_z f(\lambda,\lambda',z,E)\, d\lambda\, |d\lambda'| + O(\varepsilon^2), \\
V^+_{\mathrm{appr}}(x,E) &= i\sqrt{E}\int_{T^2} s(\lambda,\lambda')\partial_{\bar z} f(\lambda,\lambda',z,E)\, \lambda^{-2} d\lambda\, |d\lambda'| + O(\varepsilon^2), \\
a^-_z(z,E) &= -2\int_{T^2} s(\lambda,\lambda')\partial_{\bar z} f(\lambda,\lambda',z,E)\, \lambda^{-1} d\lambda\, |d\lambda'| + O(\varepsilon^2), \\
a^-_z(z,E) &= 2\int_{T^2} s(\lambda,\lambda')\partial_z f(\lambda,\lambda',z,E)\, \lambda^{-1} d\lambda\, |d\lambda'| + O(\varepsilon^2), \\
s(\lambda,\lambda') &\stackrel{\mathrm{def}}{=\!=} \operatorname{sgn}\left(\frac{1}{i}\left[\frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda}\right]\right),
\end{aligned} \tag{7.11}$$

where $x \in \mathbb{R}^2$, z is given by (3.2), and $O(\varepsilon^2)$ is considered in the uniform sense with respect to $x \in \mathbb{R}^2$.

In addition, due to (4.7) we have that

$$\begin{aligned}
\partial_z f(\lambda,\lambda',z,E) &= -i\frac{\sqrt{E}}{2}\big(\lambda^{-1} - \lambda'^{-1}\big) f(\lambda,\lambda',z,E), \\
\partial_{\bar z} f(\lambda,\lambda',z,E) &= -i\frac{\sqrt{E}}{2}\big(\lambda - \lambda'\big) f(\lambda,\lambda',z,E),
\end{aligned} \tag{7.12}$$

for each λ, $\lambda' \in T$, $z \in \mathbb{C}$, $E > 0$.

Formulas (4.2), (4.3), (4.4), (4.5), (4.6) follow immediately from (3.18), (3.19), (3.20), (7.11), (7.12).

Proposition 4.1 is proved. $\square$

Lemma 7.1. *Let $E > 0$ be fixed. Let $u(\lambda,\lambda',E)$, $(\lambda,\lambda') \in T^2$, be a complex valued function such that $u \in L^1(T^2)$ and*

$$u(\lambda,\lambda',E) = u(-\lambda',-\lambda,E), \quad (\lambda,\lambda') \in T^2. \tag{7.13}$$

Suppose that $g(p,E)$, $p=(p_1,p_2)\in\mathbb{R}^2$, $|p|\le 2\sqrt{E}$, is the function defined by the formula

$$g\Big(\sqrt{E}\,\mathrm{Re}(\lambda-\lambda'),\sqrt{E}\,\mathrm{Im}(\lambda-\lambda'),E\Big)=u(\lambda,\lambda',E), \tag{7.14}$$

where $(\lambda,\lambda')\in T^2$. Then

$$\begin{gathered}\int\limits_{|p|\le 2\sqrt{E}} e^{-ipx}g(p,E)\,dp=\frac{E}{2}\int\limits_{T^2}u(\lambda,\lambda',z,E)\left|\frac{1}{2i}\left(\frac{\lambda}{\lambda'}-\frac{\lambda'}{\lambda}\right)\right|\,|d\lambda|\,|d\lambda'|,\\ u(\lambda,\lambda',z,E)\overset{def}{=\!=}u(\lambda,\lambda',E)\times\\ \times\exp\Big(-i\frac{\sqrt{E}}{2}\big((\lambda-\lambda')\bar z+(\lambda^{-1}-\lambda'^{-1})z\big)\Big),\end{gathered} \tag{7.15}$$

where $x\in\mathbb{R}^2$, z is given by (3.2).

Actually, (7.15) arises from the following change of variables in the integration with respect to p:

$$\begin{aligned} p_1&=\sqrt{E}\,\mathrm{Re}(\lambda-\lambda')=\sqrt{E}(\cos\phi-\cos\phi'),\\ p_2&=\sqrt{E}\,\mathrm{Im}(\lambda-\lambda')=\sqrt{E}(\sin\phi-\sin\phi'),\end{aligned} \tag{7.16}$$

where $\lambda=e^{i\phi}$, $\lambda'=e^{i\phi'}$.

Proof of Proposition 4.2. Let $\lambda\in T$, $\lambda'\in T$ be defined by (3.3). It follows from (3.5), (3.6) that the following formulas are valid:

$$\begin{gathered} 2(k_1+l_1)=\sqrt{E}\big(\lambda+\lambda^{-1}+\lambda'+\lambda'^{-1}\big),\\ 2(k_2+l_2)=-i\sqrt{E}\big(\lambda-\lambda^{-1}+\lambda'-\lambda'^{-1}\big),\\ k_1\pm ik_2=\sqrt{E}\lambda^{\pm1},\quad l_1\pm il_2=\sqrt{E}\lambda'^{\pm1},\\ k_1+l_2\pm i(k_2+l_2)=\sqrt{E}\big(\lambda^{\pm1}+\lambda'^{\pm1}\big),\\ |k+l|^2=E|\lambda+\lambda'|^2,\end{gathered} \tag{7.17}$$

where $(k,l)\in M_E$.

Using Lemma 7.1 and formulas (7.17) we derive from (2.21), (2.22), (2.23), (2.27), (2.28), (2.29) the following formulas:

$$
\begin{aligned}
A^{\mathrm{div},0}_{\mathrm{appr,j}}(x,E) &= \frac{E}{2}\int\limits_{T^2} \widetilde{A}^{\mathrm{div},0}_{\mathrm{appr,j}}(\lambda,\lambda',z,E)\left|\frac{1}{2i}\left(\frac{\lambda}{\lambda'}-\frac{\lambda'}{\lambda}\right)\right| |d\lambda|\,|d\lambda'|,\\
V^{\mathrm{div},0}_{\mathrm{appr}}(x,E) &= \frac{E}{2}\int\limits_{T^2} \widetilde{V}^{\mathrm{div},0}_{\mathrm{appr}}(\lambda,\lambda',z,E)\left|\frac{1}{2i}\left(\frac{\lambda}{\lambda'}-\frac{\lambda'}{\lambda}\right)\right| |d\lambda|\,|d\lambda'|,\\
A^{\pm,0}_{\mathrm{appr},1}(x,E) &= \frac{E}{2}\int\limits_{T^2} \widetilde{A}^{\pm,0}_{\mathrm{appr},1}(\lambda,\lambda',z,E)\left|\frac{1}{2i}\left(\frac{\lambda}{\lambda'}-\frac{\lambda'}{\lambda}\right)\right| |d\lambda|\,|d\lambda'|,\\
V^{\pm,0}_{\mathrm{appr}}(x,E) &= \frac{E}{2}\int\limits_{T^2} \widetilde{V}^{\pm,0}_{\mathrm{appr}}(\lambda,\lambda',z,E)\left|\frac{1}{2i}\left(\frac{\lambda}{\lambda'}-\frac{\lambda'}{\lambda}\right)\right| |d\lambda|\,|d\lambda'|,
\end{aligned}
\tag{7.18}
$$

where $j = 1, 2$, $x \in \mathbb{R}^2$, 2, z is given by (3.2) and where

$$
\begin{aligned}
\widetilde{A}^{\mathrm{div},0}_{\mathrm{appr},1}(\lambda,\lambda',z,E) &= \frac{f^{\mathrm{lin}}(\lambda,\lambda',z,E)-f^{\mathrm{lin}}(-\lambda',-\lambda,z,E)}{4\sqrt{E}}\times\\
&\times\left(\frac{1}{\lambda^{-1}+\lambda'^{-1}}+\frac{1}{\lambda+\lambda'}\right),\\
\widetilde{A}^{\mathrm{div},0}_{\mathrm{appr},2}(\lambda,\lambda',z,E) &= \frac{f^{\mathrm{lin}}(\lambda,\lambda',z,E)-f^{\mathrm{lin}}(-\lambda',-\lambda,z,E)}{4i\sqrt{E}}\times\\
&\times\left(\frac{1}{\lambda^{-1}+\lambda'^{-1}}-\frac{1}{\lambda+\lambda'}\right),\\
\widetilde{V}^{\mathrm{div},0}_{\mathrm{appr}}(\lambda,\lambda',z,E) &= \frac{f^{\mathrm{lin}}(\lambda,\lambda',z,E)+f^{\mathrm{lin}}(-\lambda',-\lambda,z,E)}{2},\\
\widetilde{A}^{\pm,0}_{\mathrm{appr},1}(\lambda,\lambda',z,E) &= \frac{1}{2\sqrt{E}}\frac{f(\lambda,\lambda',z,E)-f(-\lambda',-\lambda,z,E)}{\lambda^{\pm1}+\lambda'^{\pm1}},\\
\widetilde{V}^{\pm,0}_{\mathrm{appr}}(\lambda,\lambda',z,E) &= \frac{\lambda'^{\pm1}f^{\mathrm{lin}}(\lambda,\lambda',z,E)+\lambda^{\pm1}f^{\mathrm{lin}}(-\lambda',-\lambda,z,E)}{\lambda^{\pm1}+\lambda'^{\pm1}},
\end{aligned}
\tag{7.19}
$$

where λ, $\lambda' \in T$, $z \in \mathbb{C}$, $f^{\mathrm{lin}}(\lambda,\lambda',z,E)$ is defined according to (4.7).

Now using (7.19) we represent each integral of (7.18) as a sum of the integral containing $f^{\mathrm{lin}}(\lambda,\lambda',z,E)$ and the integral containing $f^{\mathrm{lin}}(-\lambda',-\lambda,z,E)$ within the integrands.

Making the change of variables $(\lambda,\lambda') \to (-\lambda',-\lambda)$ in each integral containing $f^{\mathrm{lin}}(-\lambda',-\lambda,z,E)$ and taking into account (2.27) for $\widehat{A^{\pm}_2}$ we obtain formulas (4.8), (4.9). $\square$

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